Torsion pairs via large silting mutation

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Stp. Throughout, we will assume that

all cats. are additive;

all subcategories are full and iso-closed additive subcategories;

all arbitrary $\prod$'s and $\coprod$'s are set-indexed;

$(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an ET-cat. with positive extensions;

$(\mathscr{T}, [1], \Delta)$ is a Δ'd-cat. with $\prod$'s, seen as an ET-cat. “ ”;

Λ is a finite-dimensional algebra over a field $\mathbb{k} = \overline{\mathbb{k}}$;

R is an associative unital ring;

Ntt.

D(R) is the unbounded derived category of Mod(R), and we often identify the latter with the complexes in D(R) concentrated in degree 0.

For $\mathcal{S} \subseteq \mathscr{C}$,

$\mathcal{S}^{\wedge}$ (respectively, $\mathcal{S}^{\vee}$) denotes the class of all objects in $\mathscr{C}$ which have a finite $\mathcal{S}$-(co)resolution;

Prod($\mathcal{S}$) (respectively, Add($\mathcal{S}$)) is the closure of $\mathcal{S}$ under (co)products and summands in $\mathscr{C}$.

Section 1

Introduction

Motivation

- A torsion pair (TP) in $\mathrm{mod}(\Lambda)$ decomposes $\mathrm{mod}(\Lambda)$ into simpler subcategories in a unique way while preserving homological information.
- By [AIR14][DF15], compact 2-term silting complexes over Λ are 1-1 with *functorially finite* TPs in $\mathrm{mod}(\Lambda)$ (denoted $\text{f-tors}(\Lambda)$).
- The former are mutable at each indec. summand $\rightsquigarrow$ (nice) mutation on $\text{f-tors}(\Lambda)$!
- By [Cra94][BP17][ZW17][Ang18], TPs in $\mathrm{mod}(R)$ (denoted $\mathrm{tors}(R)$) are 1-1 with (**large**) 2-term **cosilting** complexes over R.

Qst. *How can we mutate 2-term cosilting complexes?*

Rmk. Different approaches to this question include [ALS24][ALS26][Ang26]; we develop a direct approach in the *category of large injective copresentations* $\mathcal{K}^{[0,1]}(\mathrm{Inj}\,R)$ — which contains all 2-term cosilting complexes over R.

Cosilting mutation (for objects)

In [Ang+25], *cosilting mutation* is defined for cosilting objects via HRS-tilts.

Prp-Dfn. [PV18][HRS96][Ang+25]

(a) An object $\omega \in \mathscr{T}$ is *cosilting* if $\mathbb{T}_\omega = ({}^{\perp_{\leq 0}}\omega, \omega^{\perp_{>0}})$ is a t-structure in $\mathscr{T}$. In this case, the *presilting* subcategory $\mathrm{Prod}(\omega)$ of $\mathscr{T}$ determines $\mathbb{T}_\omega$.

(b) The *left HRS-tilt* of a t-structure $(\mathcal{X}, \mathcal{Y})$ in $\mathscr{T}$ at a torsion pair $(\mathcal{T}, \mathcal{F})$ in its heart $\mathcal{X}[-1] \cap \mathcal{Y}$ is a t-structure given by

$$(\mathcal{X}[1] * \mathcal{T}[1], \mathcal{F}[1] * \mathcal{Y}).$$

The *right HRS-tilt* is defined dually.

(c) Let $\omega, \omega' \in \mathscr{T}$ be cosilting objects and $\mathcal{D} = \mathrm{Prod}(\omega) \cap \mathrm{Prod}(\omega')$. We call ω' a *left mutation* of ω (w.r.t. $\mathcal{D}$) if there exists a distinguished triangle

$$\omega \xrightarrow{f} B_0 \to B_1 \to \omega[1]$$

with f a left $\mathcal{D}$-approx. and $\mathrm{Prod}(B_0 \bigoplus B_1) = \mathrm{Prod}(\omega')$ (dually for *right*).

Rmk. The cosilting mutation of a 2-term cosilting complex is once again a cosilting complex, but it is not 2-term in general ($\not\rightsquigarrow$ mutation on $\mathrm{tors}(R)$!).

Silting mutation (for compact complexes)

Prp-Dfn. [AI12] Let $\sigma = \bigoplus_{i=1}^{n} \sigma_i \in \mathcal{K}^b(\mathrm{proj}\Lambda)$ be a basic silting complex. Then each indecomposable summand $\sigma_j \in \mathrm{add}(\sigma)$ has a right $\mathrm{add}(\bigoplus_{i=1, i\neq j}^{n} \sigma_i)$-approximation. We consider the corresponding distinguished triangle

$$\sigma_j[-1] \to N_{\sigma_j} \to \bigoplus_{i=1, i\neq j}^{n} \sigma_i \to \sigma_j,$$

and define the *right mutation* of σ at j, which is a silting complex, as

$$\left(\bigoplus_{i=1, i\neq j}^{n} \sigma_i \right) \bigoplus N_{\sigma_j}.$$

The *left mutation* of σ at j is defined dually.

Silting mutation (for subcategories) I

Dfn. Let $\mathcal{S} \subseteq \mathscr{C}$.

(a) We denote the closure of $\mathcal{S}$ under cones, cocones, extensions and summands in $\mathscr{C}$ by $\operatorname{thick}(\mathcal{S})$.

(b) For $I \subseteq \mathbb{Z}_{\geq 0}$, we define

$$^{\perp_I}\mathcal{S} := \{C \in \mathscr{C} \mid \mathbb{E}^i(C, S) = 0 \text{ for all } S \in \mathcal{S} \text{ and } i \in I\}.$$

$\mathcal{S}^{\perp_I}$ is defined dually.

(c) We say that $\mathcal{S}$ is a *presilting* subcategory of $\mathscr{C}$ if $\operatorname{smd}(\mathcal{S}) = \mathcal{S} \subseteq {}^{\perp_{>0}}\mathcal{S}$. In addition, if $\operatorname{thick}(\mathcal{S}) = \mathscr{C}$, then it is a *silting* subcategory of $\mathscr{C}$.

(d) A *cotorsion pair* (CP) in $\mathscr{C}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathscr{C}$ such that $\mathcal{X} = {}^{\perp_1}\mathcal{Y}$ and $\mathcal{X}^{\perp_1} = \mathcal{Y}$. Furthermore, it is

- (B) *bounded* if $\mathcal{X}^{\wedge} = \mathscr{C} = \mathcal{Y}^{\vee}$;
- (H) *hereditary* if $\mathcal{X} \subseteq {}^{\perp_{>0}}\mathcal{Y}$;
- (C) *complete* if, for each $C \in \mathscr{C}$, there exist $\mathfrak{s}$-conflations $C \to Y \to X$ and $Y' \to X' \to C$ with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$.

Silting mutation (for subcategories) II

Thm. (Bijections) [AT22][Men+13] There exist mutually inverse bijections

$$\begin{array}{ccc} (\mathcal{X},\mathcal{Y}) & \longmapsto & \mathcal{X}\cap\mathcal{Y} \\ \mathrm{BHCCPs}(\mathscr{C}) & \rightleftarrows & \mathrm{silt}(\mathscr{C}). \\ (\mathcal{S}^{\vee},\mathcal{S}^{\wedge}) & \longleftarrow & \mathcal{S} \end{array}$$

Prp-Dfn. (Partial order) [AT22] For $\mathcal{M},\mathcal{N} \in \mathrm{silt}(\mathscr{C})$, denote by $\mathcal{M} \geqslant \mathcal{N}$ the condition:

$$\mathbb{E}^n(\mathcal{M},\mathcal{N}) = 0 \text{ for all } n > 0.$$

TFCAE.

(a) $\mathcal{M} \geqslant \mathcal{N}$.

(b) $\mathcal{M}^{\wedge} \supseteq \mathcal{N}^{\wedge}$.

(c) $\mathcal{M}^{\vee} \subseteq \mathcal{N}^{\vee}$.

In particular, $\geqslant$ is a partial order on $\mathrm{silt}(\mathscr{C})$.

Lmm. (Maximality property) [AT22] Let $\mathcal{M} \in \mathrm{silt}(\mathscr{C})$. If $\mathcal{N} \subseteq \mathscr{C}$ is such that $\mathcal{M} \subseteq \mathcal{N} \subseteq {}^{\perp_{>0}}\mathcal{N}$, then $\mathcal{M} = \mathcal{N}$.

Silting mutation (for subcategories) III

Lmm. [AT25] Let $\mathcal{D} \subseteq \mathcal{S} \subseteq \mathscr{C}$ with $\mathcal{S} \subseteq {}^{\perp_{>0}}\mathcal{S}$. Suppose that there exists an $\mathfrak{s}$-conflation $S \xrightarrow{f} D \xrightarrow{g} N$ with $S \in \mathcal{S}$ and f a left $\mathcal{D}$-approx. Then

f is a left $\mathcal{D}$-approx. of S with $N \in {}^{\perp_{>0}}\mathcal{D}$;
g is a right $\mathcal{D}$-approx. of N with $S \in \mathcal{D}^{\perp_{>0}}$;
$N \in \mathcal{S}^{\perp_{>0}} \cap {}^{\perp_{>1}}\mathcal{S}$.

Dfn. [AT25] Let $\mathcal{D} \subseteq \mathcal{S} \subseteq \mathscr{C}$. $\mathcal{D}$ is a *good contravariantly finite subcategory* of $\mathcal{S}$ if each object in $\mathcal{S}$ has a right $\mathcal{D}$-approx. which is an $\mathfrak{s}$-deflation. If $\mathcal{S}$ is a presilting subcategory of $\mathscr{C}$ then, for each $S \in \mathcal{S}$, we consider the corresponding $\mathfrak{s}$-conflation

$$N_S \to D \xrightarrow{g} S,$$

and define the *right mutation* of $\mathcal{S}$ w.r.t. $\mathcal{D}$ as

$$\mu^R(\mathcal{S}; \mathcal{D}) := \operatorname{add}(\mathcal{D} \cup \{N_S\}_{S \in \mathcal{S}}).$$

The *left mutation* $\mu^L(\mathcal{S}; \mathcal{D})$ of $\mathcal{S}$ w.r.t. a *good covariantly finite* subcategory $\mathcal{D}$ of $\mathcal{S}$ is defined dually.

Rmk. [AT25] These do not depend on particular choices of approximations!

(Co)silting characterizations

Rmk. [AI12][Gar26][Ang19] Let σ be a complex of projective modules.

(1) σ is a compact silting complex over Λ iff $\mathrm{add}(\sigma) \in \mathrm{silt}\big(\mathcal{K}^b(\mathrm{proj}\Lambda)\big)$.

(2) σ is a 2-term "   " iff $\mathrm{add}(\sigma) \in \mathrm{silt}\big(\mathcal{K}^{[-1,0]}(\mathrm{proj}\Lambda)\big)$.

(3) σ is a large silting complex over R iff $\mathrm{Add}(\sigma) \in \mathrm{silt}\big(\mathcal{K}^b(\mathrm{Proj}R)\big)$.

Prp. [BN26] Let $\omega \in \mathrm{D}(R)$. Then

(a) ω is a cosilting complex over R iff $\mathrm{Prod}(\omega) \in \mathrm{silt}\big(\mathcal{K}^b(\mathrm{Inj}R)\big)$.

(b) ω is a 2-term cosilting complex over R iff $\mathrm{Prod}(\omega) \in \mathrm{silt}\big(\mathcal{K}^{[0,1]}(\mathrm{Inj}R)\big)$.

Qst. *How can we mutate $\prod$-closed silting subcategories of $\mathcal{K}^{[0,1]}\mathrm{Inj}(R)$?*

Rmk. Note that $\mathcal{K}^{[0,1]}\mathrm{Inj}(R)$:

is *not* shift-closed in $\mathrm{D}(R)$ $\rightsquigarrow$ it is **not** a triangulated subcategory;

is extension-closed in $\mathrm{D}(R)$ $\rightsquigarrow$ it **is** an extriangulated category.

Section 2

Large silting mutation

Large silting mutation (for $\prod$-closed subcategories) I

Stp. Suppose that $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has *exact products* (i.e., that $\mathfrak{s}$-conflations are stable under taking degree-wise arbitrary products).

Dfn. [BN26] Let $\mathcal{D} \subseteq \mathcal{S} \subseteq \mathscr{C}$ with $\mathcal{D}$ a good contravariantly finite subcategory of $\mathcal{S}$. If $\mathcal{S}$ is a $\prod$-closed presilting subcategory of $\mathscr{C}$ then, for each $S \in \mathcal{S}$, we consider the corresponding $\mathfrak{s}$-conflation

$$N_S \to D \xrightarrow{g} S,$$

and define the *right product-mutation* of $\mathcal{S}$ w.r.t. $\mathcal{D}$ as

$$\prod^R(\mathcal{S};\mathcal{D}) := \mathrm{Prod}(\mathcal{D} \cup \{N_S\}_{S\in\mathcal{S}}).$$

The *left product-mutation* $\prod^L(\mathcal{S};\mathcal{D})$ of $\mathcal{S}$ w.r.t. a good covariantly finite subcategory $\mathcal{D}$ of $\mathcal{S}$ is defined dually.

Rmk. [BN26] Neither product-mutation depends on particular choices of approximations.

Large silting mutation (for $\prod$-closed subcategories) II

Lmm. [BN26] Let $\mathcal{D}, \mathcal{S}$ be $\prod$-closed subcategories of $\mathscr{C}$. If $\mathcal{D}$ is a good contravariantly finite subcategory of $\mathcal{S}$, then it is a good covariantly finite subcategory of $\prod^R(\mathcal{S}; \mathcal{D})$.

Thm. [BN26][AT25] Let $\mathcal{S} = \mathrm{Prod}(\mathcal{S})$ be a presilting subcategory of $\mathscr{C}$ and $\mathcal{D} = \mathrm{Prod}(\mathcal{D})$ be a good contravariantly finite subcategory of $\mathcal{S}$.

(a) $\prod^R(\mathcal{S}; \mathcal{D})$ is a product-closed presilting subcategory of $\mathscr{C}$.

(b) $\prod^R(\mathcal{S}; \mathcal{D}) \geqslant \mathcal{S}$, where the equality holds iff $\mathcal{S} = \mathcal{D}$.

(c) If $\mathcal{S}$ is a silting subcategory of $\mathscr{C}$, then so is $\prod^R(\mathcal{S}; \mathcal{D})$.

(d) $\mathcal{D}$ is a good covariantly finite subcategory of $\prod^R(\mathcal{S}; \mathcal{D})$ and

$$\prod^L\left(\prod^R(\mathcal{S}; \mathcal{D}); \mathcal{D}\right) = \mathcal{S}.$$

Qst. *Can we use this to mutate objects in the category?*

Large silting mutation (for *producers*) I

Dfn. [BN26][Ang+25]

(a) Let $\mathcal{S}$ be a presilting subcategory of $\mathscr{C}$. A *producer* of $\mathcal{S}$ is an object $C \in \mathscr{C}$ such that $\mathcal{S} = \operatorname{Prod}(C)$.

(b) Let $C, C' \in \mathscr{C}$ be producers of silting subcategories of $\mathscr{C}$ and $\mathcal{D} = \operatorname{Prod}(C) \cap \operatorname{Prod}(C')$. We call C a *right mutation* of C' (w.r.t. $\mathcal{D}$) if there exists an $\mathfrak{s}$-conflation

$$A_1 \to A_0 \xrightarrow{g} C'$$

with g a right $\mathcal{D}$-approx. and $\operatorname{Prod}(A_1 \bigoplus A_0) = \operatorname{Prod}(C)$ (dually for *left*).

Large silting mutation (for *producers*) II

Stp. Suppose that

- there exists an $\mathbb{E}$-injective cogenerator Q (i.e., an object $Q \in \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})$ such that, for all $C \in \mathscr{C}$, there exist a set J and an $\mathfrak{s}$-inflation $C \to Q^J$);
- $\mathrm{Prod}(X^{\perp_{>0}}) = X^{\perp_{>0}}$ for all $X \in \mathscr{C}$.

Lmm. [BN26] Let $C \in \mathscr{C}$ be a producer of a silting subcategory of $\mathscr{C}$, $\mathcal{D} \subseteq \mathscr{C}$ be such that $\mathrm{Prod}(\mathcal{D}) \subseteq \mathrm{Prod}(C)$, and $A_1 \to A_0 \xrightarrow{g} C$ be an $\mathfrak{s}$-conflation such that g is a right $\mathrm{Prod}(\mathcal{D})$-approximation. Then $\mathrm{Prod}(A_1 \bigoplus A_0) \in \mathrm{silt}(\mathscr{C})$.

Thm. [BN26] Let $C, C' \in \mathscr{C}$ be producers of silting subcategories of $\mathscr{C}$ and $\mathcal{D} = \mathrm{Prod}(C) \cap \mathrm{Prod}(C')$. TFCAE.

(a) C' is a left mutation of C.

(b) C is a right mutation of C'.

(c) $\mathcal{D}$ is a good contravariantly finite subcategory of $\mathrm{Prod}(C')$ and $\prod^R(\mathrm{Prod}(C'), \mathcal{D}) = \mathrm{Prod}(C)$.

(d) $\mathcal{D}$ is a good covariantly finite subcategory of $\mathrm{Prod}(C)$ and $\prod^L(\mathrm{Prod}(C), \mathcal{D}) = \mathrm{Prod}(C')$.

Large silting bijections I

Qst. *When do all $\prod$-closed silting subcategories of $\mathscr{C}$ have a producer?*

Stp. Suppose that all objects in $\mathscr{C}$ have finite $\mathbb{E}$-injective dimension.

Lmm. [BN26] Let $(\mathcal{X}, \mathcal{Y}) \in \mathrm{BHCCP}(\mathscr{C})$ be such that $\mathrm{Prod}(\mathcal{X} \cap \mathcal{Y}) = \mathcal{X} \cap \mathcal{Y}$. Note that $\mathcal{Y} = (\mathcal{X} \cap \mathcal{Y})^\wedge$ by a previous Theorem and consider a finite $(\mathcal{X} \cap \mathcal{Y})$-resolution of the $\mathbb{E}$-injective cogenerator Q

$$\begin{aligned} L_1 &\to Z_0 \to Q, \\ L_2 &\to Z_1 \to L_1, \\ L_3 &\to Z_2 \to L_2, \\ &\vdots \\ 0 &\to Z_m \to L_m. \end{aligned}$$

Then

$$\mathcal{X} \cap \mathcal{Y} = \mathrm{Prod}\left(\bigoplus_{i=0}^{m} Z_i \right).$$

Large silting bijections II

Thm. [BN26] There exist mutually inverse bijections

$$\begin{array}{c}
\left\{\text{producers of } \prod\text{-closed silting subcategories of } \mathscr{C}\right\}_{/\sim_{\mathrm{Prod}}} \\
\begin{array}{ccc}
[C] & & \\
\big\downarrow & \downarrow\!\!\uparrow & \Theta \\
\mathrm{Prod}(C) & &
\end{array} \\
\{\mathcal{S} \in \mathrm{silt}(\mathscr{C}) \mid \mathrm{Prod}(\mathcal{S}) = \mathcal{S}\} \\
\begin{array}{ccc}
\mathcal{S} & & \mathcal{X}\cap\mathcal{Y} \\
\big\downarrow & \downarrow\!\!\uparrow & \big\uparrow \\
(\mathcal{S}^{\vee},\mathcal{S}^{\wedge}) & & (\mathcal{X},\mathcal{Y})
\end{array} \\
\{(\mathcal{X},\mathcal{Y}) \in \mathrm{BHCCP}(\mathscr{C}) \mid \mathrm{Prod}(\mathcal{X}\cap\mathcal{Y}) = \mathcal{X}\cap\mathcal{Y}\},
\end{array}$$

where $\Theta(\mathcal{S})$ is calculated using the previous Lemma.

Large silting bijections II

Thm. [BN26] There exist mutually inverse bijections

$$\left\{\text{producers of } \prod\text{-closed silting subcategories of } \mathscr{C}\right\}_{/\sim_{\mathrm{Prod}}}$$

$$[C] \longmapsto \mathrm{Prod}(C) \qquad \downarrow\uparrow\ \Theta$$

$$\{\mathcal{S} \in \mathrm{silt}(\mathscr{C}) \mid \mathrm{Prod}(\mathcal{S}) = \mathcal{S}\}$$

$$\mathcal{S} \longmapsto (\mathcal{S}^{\vee}, \mathcal{S}^{\wedge}) \qquad \downarrow\uparrow \qquad \mathcal{X}\cap\mathcal{Y} \longleftarrow (\mathcal{X}, \mathcal{Y})$$

$$\{(\mathcal{X}, \mathcal{Y}) \in \mathrm{BHCCP}(\mathscr{C}) \mid \mathrm{Prod}(\mathcal{X}) = \mathcal{X}\},$$

where $\Theta(\mathcal{S})$ is calculated using the previous Lemma.

Large silting bijections II

Thm. [BN26] There exist mutually inverse bijections

$$\left\{\text{producers of } \prod\text{-closed silting subcategories of } \mathscr{C}\right\}_{/\sim_{\mathrm{Prod}}}$$

$$[C] \longmapsto \mathrm{Prod}(C) \qquad \Big\downarrow \Big\uparrow \Theta$$

$$\{\mathcal{S} \in \mathrm{silt}(\mathscr{C}) \mid \mathrm{Prod}(\mathcal{S}) = \mathcal{S}\}$$

$$\mathcal{S} \longmapsto (\mathcal{S}^{\vee}, \mathcal{S}^{\wedge}) \qquad \Big\downarrow \Big\uparrow \qquad \mathcal{X}\cap\mathcal{Y} \longmapsfrom (\mathcal{X}, \mathcal{Y})$$

$$\{(\mathcal{X}, \mathcal{Y}) \in \mathrm{HCCP}(\mathscr{C}) \mid \mathrm{Prod}(\mathcal{X}) = \mathcal{X} \text{ and } \mathcal{X}^{\wedge} = \mathscr{C}\},$$

where $\Theta(\mathcal{S})$ is calculated using the previous Lemma.

Duality

Rmk. If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has instead

exact coproducts;

an $\mathbb{E}$-projective generator;

$\mathrm{Add}({}^{\perp_{>0}}X) = {}^{\perp_{>0}}X$ for all $X \in \mathscr{C}$;

then we can similarly define coproduct-mutations $\coprod^L(\mathcal{S}; \mathcal{D})$ and $\coprod^R(\mathcal{S}; \mathcal{D})$, and obtain analogous results for $\coprod$-closed (pre)silting subcategories of $\mathscr{C}$!

Section 3

Large injective copresentations

The category $\mathcal{K}^{[0,1]}(\mathrm{Inj}R)$

Rmk. Note that $\mathcal{K}^{[0,1]}(\mathrm{Inj}R)$

[NP19] has an extriangulated structure by restriction of $\mathrm{D}(R)$;

[Gar26][LN19][GNP21] has both enough $\mathbb{E}$-projectives ($\mathrm{Inj}(R)[-1]$) and $\mathbb{E}$-injectives ($\mathrm{Inj}(R)$) $\rightsquigarrow$ positive extensions via syzygies and cosyzigies—in particular, $\mathbb{E}^2 = 0 \rightsquigarrow$ finite positive global dimension;

[Nee14] has exact products induced by those in $\mathrm{D}(R)$;

[BN26] has an $\mathbb{E}$-injective generator $Q = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$;

[AMP25] has, for each $X \in \mathscr{C}$ and family $\{C_j\}_{j\in J}$ of objects in $\mathscr{C}$, that

$$\mathbb{E}\Big(X, \prod_{j\in J} C_j\Big) \hookrightarrow \prod_{j\in J} \mathbb{E}(X, C_j),$$

by which $\mathrm{Prod}(X^{\perp_{>0}}) = X^{\perp_{>0}}$.

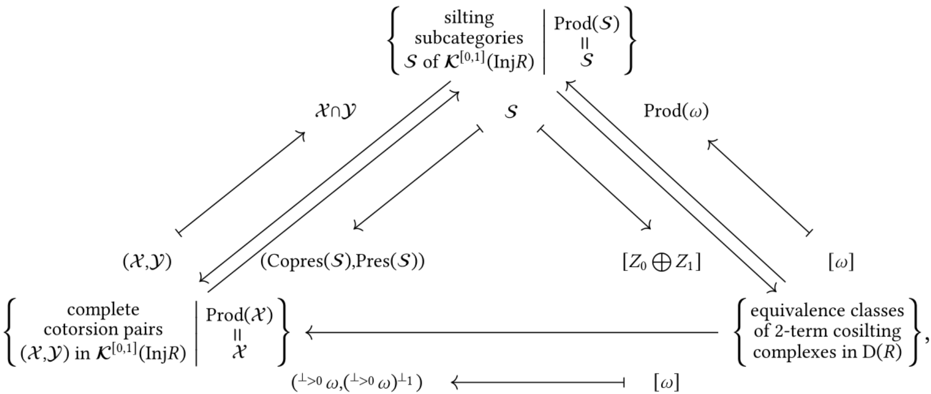
Thus, *all previous results apply* to $\mathcal{K}^{[0,1]}(\mathrm{Inj}R)$! (More generally, they apply to

$$\mathrm{Prod}(\omega)^{[-m,0]} = \mathrm{Prod}(\omega)[-m] * \mathrm{Prod}(\omega)[-(m-1)] * \cdots * \mathrm{Prod}(\omega)$$

for any cosilting object ω in $(\mathscr{T}, [1], \Delta)$. $\mathrm{Prod}(\omega)^{[-m,0]}$ is $(m-1)$-*Auslander*.)

2-term cosilting complexes

Cor. [BN26] There exists a commutative diagram of bijections



where Z_0 and Z_1 are the extensions appearing in the $\mathcal{S}$-presentation of any injective cogenerator Q of $\mathrm{Mod}(R)$ (as per the previous Lemma and Remark).

Other (co)producers of large silting subcategories

Exp. [BN26, §4.1 & §4.3]

(1) $(n+1)$-term (co)silting complexes relative to arbitrary (co)silting objects, including (the duals of) connective dg algebras (c.f. [AMY19]).

(2) Infinite-dimensional n-(co)tilting modules over a ring of finite global dimension.

(?) "Bounded infinite-dimensional cluster-tilting objects"?

Section 4

Mutability

Partial results

(1) When $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has an $\mathbb{E}$-injective cogenerator Q, we may obtain a criterion for Bongartz completions in terms of approximations of Q, similarly to [GNP23].

(2) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has *exact product-inclusions*, then it has *universal $\mathbb{E}$-coextensions* (see [AMP25]). These can be used to construct Bongartz completions, analogously to [AT25].

(3) Further work: connection to extriangulated HRS-tilts in [AET23][WZ26].

(4) Further work: connection to (co)silting discreteness in [AMY19].

(5) Further work: seek large silting phenomena (next slide) in the work of Angeleri Hügel-Laking-Pfeifer.

Large silting phenomenon

Rmk. In general, producers of $\prod$-closed silting subcategories cannot be decomposed into indecomposables. Moreover, they may contain "redundant" (indecomposable) summands, which may have useful approximation properties.

Exp. Let A be the Kronecker algebra over $\mathbb{k} = \overline{\mathbb{k}}$ and P be a subset of the projective line over $\mathbb{k}$. By [BK03], large (1-)cotilting modules are of the form

$$C_P = \left(\prod_{x \in P} S_x[-\infty]\right) \bigoplus G \bigoplus \left(\prod_{y \notin P} S_y[\infty]\right)$$

up to equivalence, where $\mathrm{Prod}(G) \subseteq \mathrm{Prod}(S_x[\infty])$ for any $x \in \mathbb{X}$ by [Rin98]. Moerover, for each $x \in \mathbb{X}$, there exist a set J_x and a short exact sequence

$$0 \to S_x[-\infty] \xrightarrow{f_x} G^{J_x} \xrightarrow{g_x} S_x[\infty] \to 0$$

with f_x a left $\mathrm{Prod}(G)$-approximation of $S_x[-\infty]$ and g_x a right $\mathrm{Prod}(G)$-approximation of $S_x[\infty]$. Using these as "building blocks", the "redundant" indecomposable G enables right and left mutations of C_p!

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