Torsion pairs via large silting mutation

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Mathematics in Conversation

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Outline

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2. Mutations
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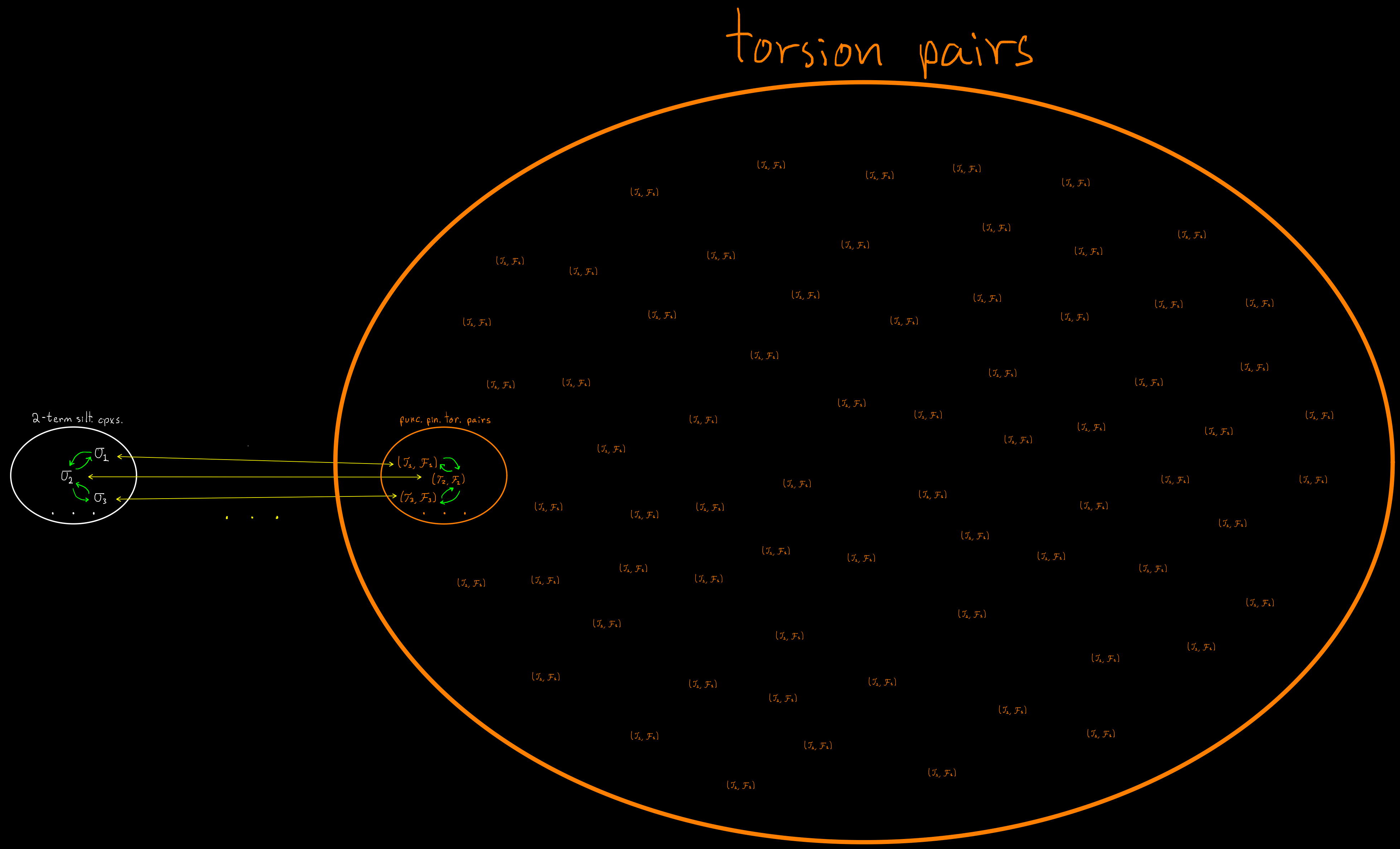
1. Motivation



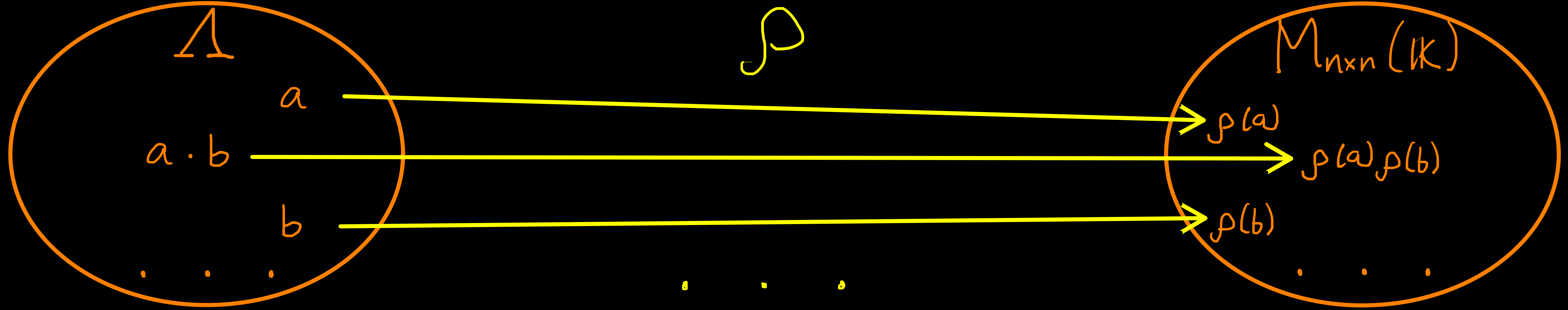
Representation Theory of Algebras

Stp. Let $\mathbb{K}$ be a field and $n \in \mathbb{Z}_{>0}$.

• A $\mathbb{K}$-*algebra* Λ is an algebraic structure akin to a $\mathbb{K}$-vector space which is also a ring.

Exp. $M_{n\times n}(\mathbb{K})$ is a $\mathbb{K}$-algebra (in particular, $n=1$ recovers $\mathbb{K}$!).

• Their highly complex structures are studied approximately via their *representations*.

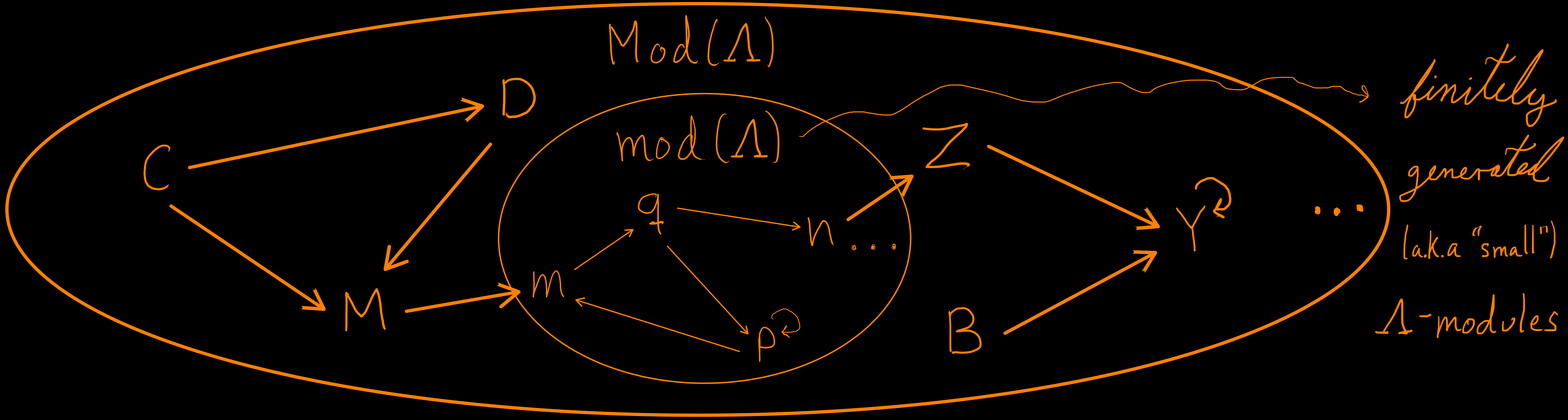


• Any given Λ may have infinitely many representations, each partially modelling its structure!

Small and large modules

• The representations of a given Λ are also called *Λ-modules*, and they come in two "sizes".

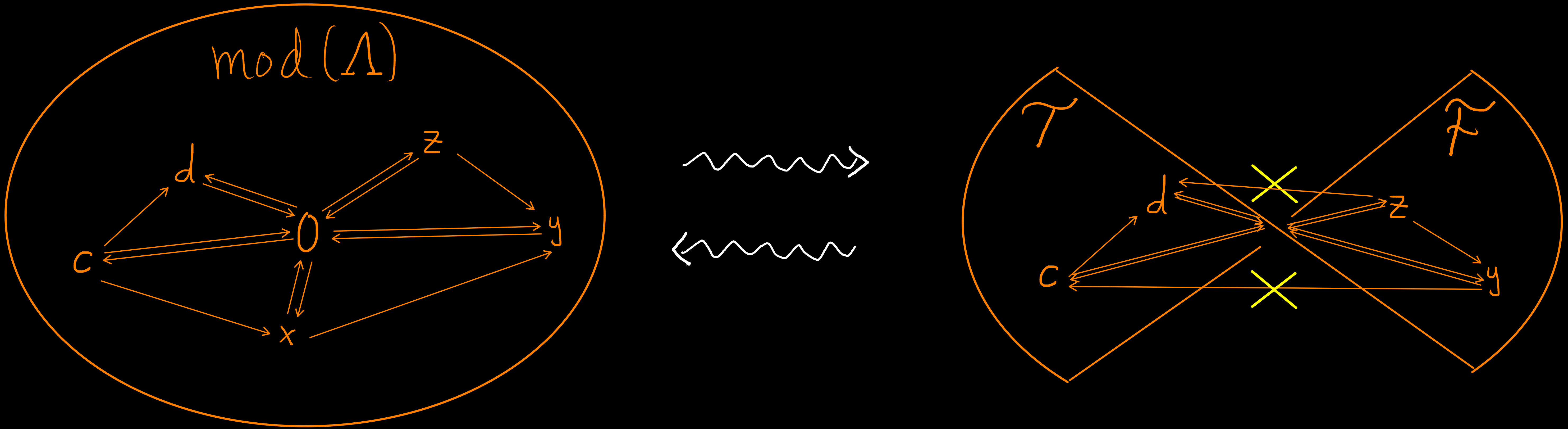
Exp. $\mathbb{K}$-modules = $\mathbb{K}$-vector spaces (of finite/infinite dimension)



• A powerful approach in the RToA is to study the structures of these module categories.

Torsion pairs

- A *torsion pair* $(\mathcal{T},\mathcal{F})$ is a <u>unique</u> decomposition of a category in two "disjoint" subcategories from which it can be reconstructed (loose analogy: orthogonal complement and $\oplus$ in linear alg.).

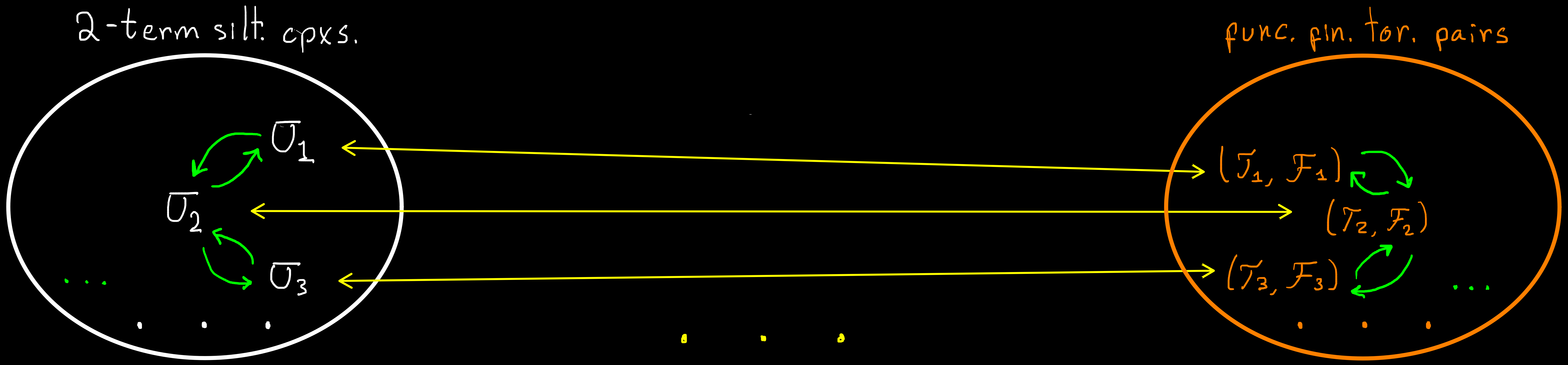


- Idea: Control torsion pairs in mod(Λ) $\rightsquigarrow$ study mod(Λ) $\rightsquigarrow$ understand Λ!

Induced mutations on torsion pairs

- [Adachi-Iyama-Reiten '14] [Derksen-Fei '15] A certain subclass of torsion pairs in $\mathrm{mod}(\Lambda)$ is in bijection with the class of "(small) 2-term silting complexes" over Λ.

- The latter class has a well-defined operation of *silting mutation*, which induces a mutation on the corresponding subclass of torsion pairs in $\mathrm{mod}(\Lambda)$.



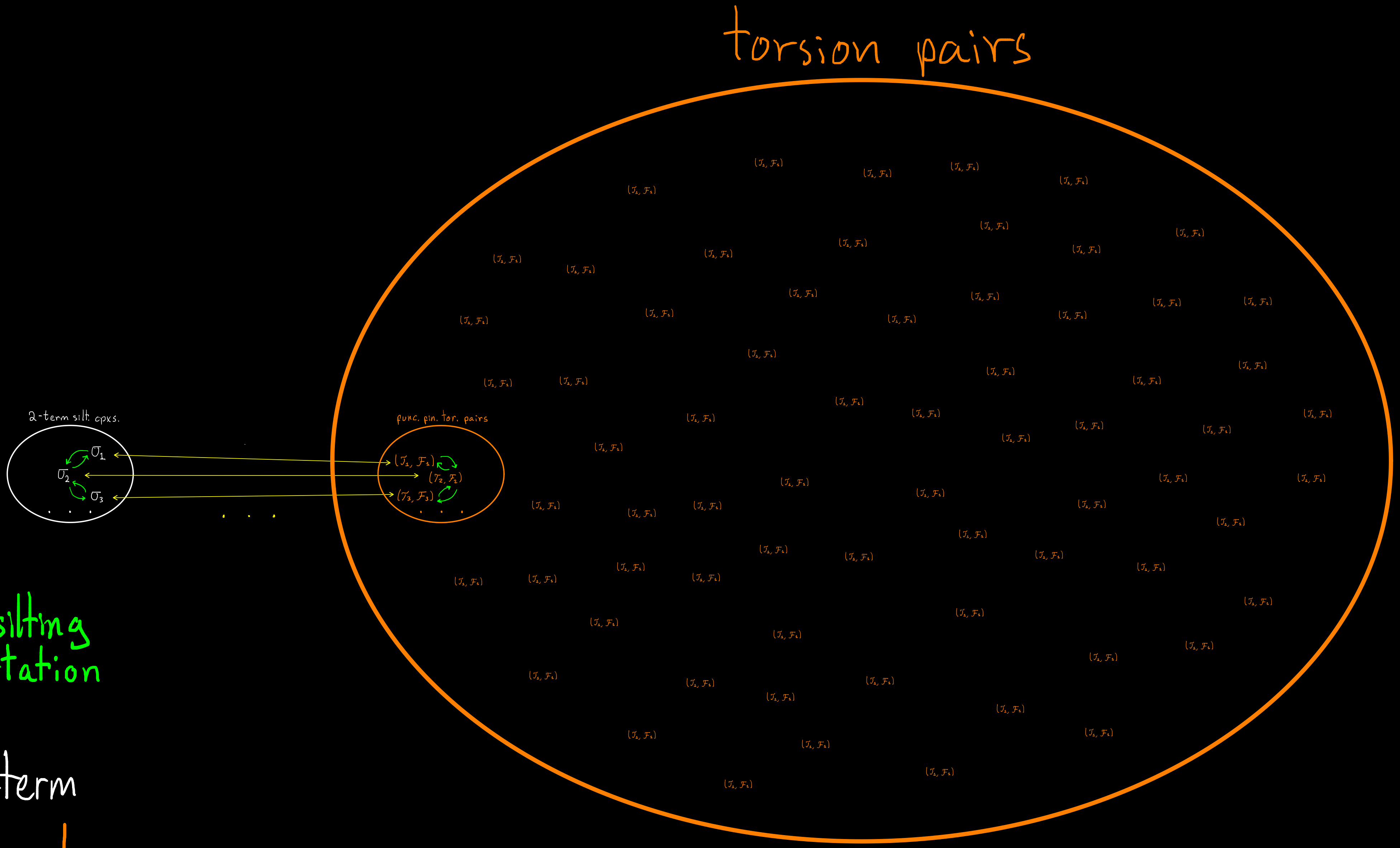
Motivating question

Qst. Can we do something analogous for *all* torsion pairs in $\mathrm{mod}(\Lambda)$?

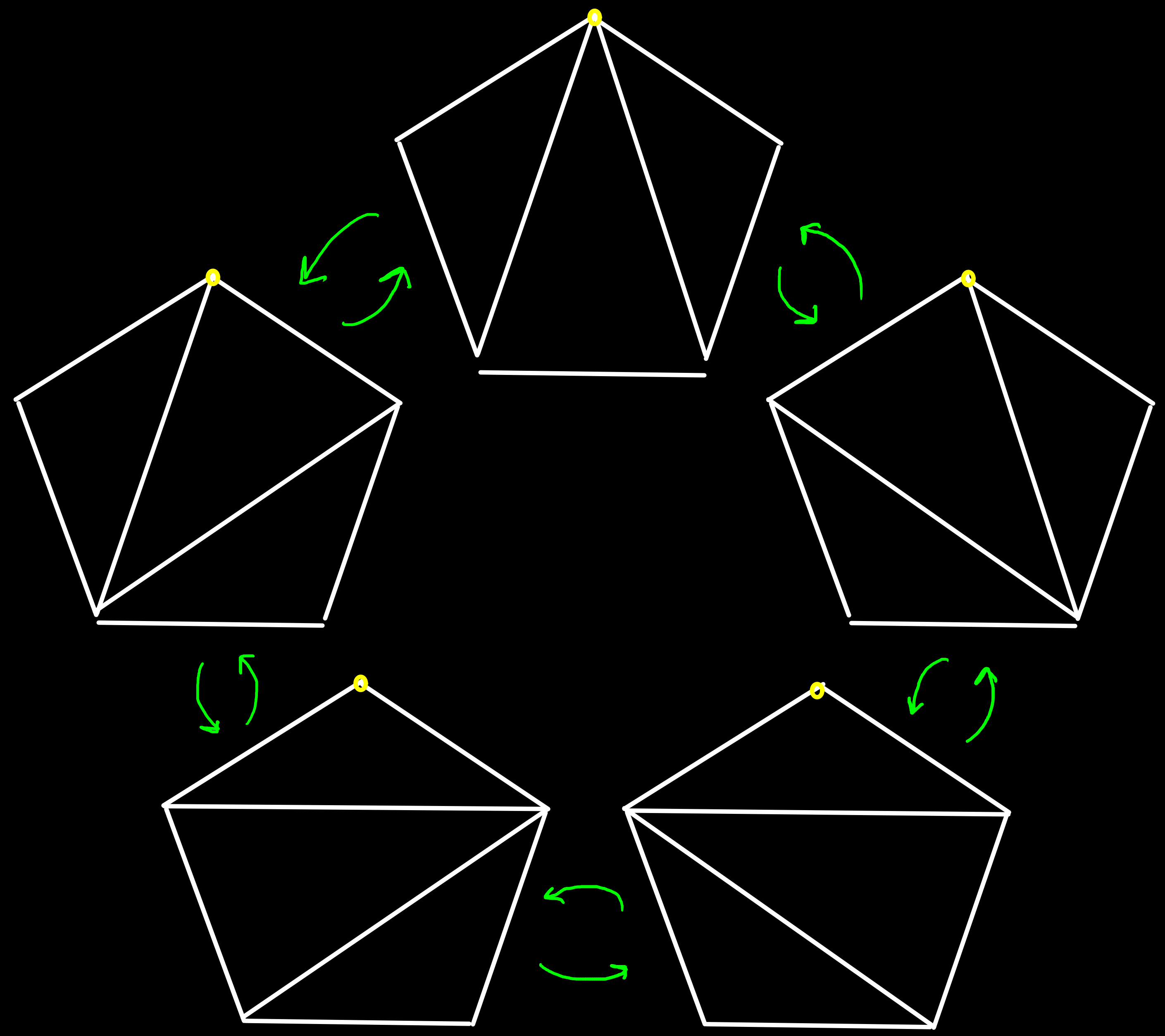
$$\{\text{torsion pairs}\}$$
$$\Big\updownarrow 1:1$$
$$\{\text{2-term cosilting complexes}\}$$
$$\big\downarrow$$
$$\{\text{cosilting objects}\}$$

cosilting mutation

The cosilting mutation of a 2-term cos. cpx. is not necessarily 2-term!



2. Mutations



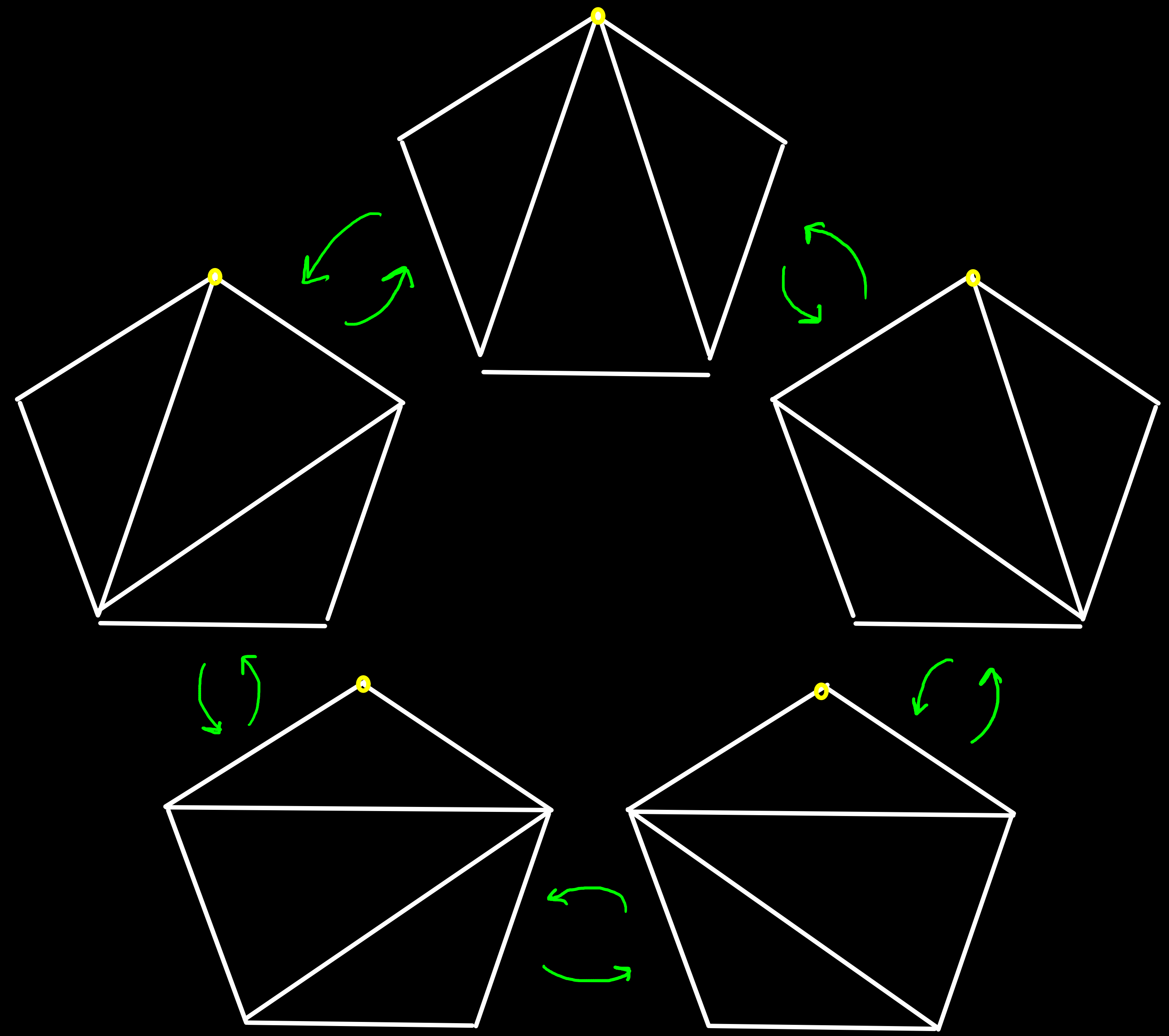
Combinatorial model

Consider the following maximal sets of non-crossing diagonals (a.k.a. triangulations).

Note that:

- Any diagonal in a triangulation can be flipped, resulting in another triangulation.
- Each one of these flips has an inverse.

In essence, this is how 2-term silt. cpxs. behave under silting mutation!



Conflations

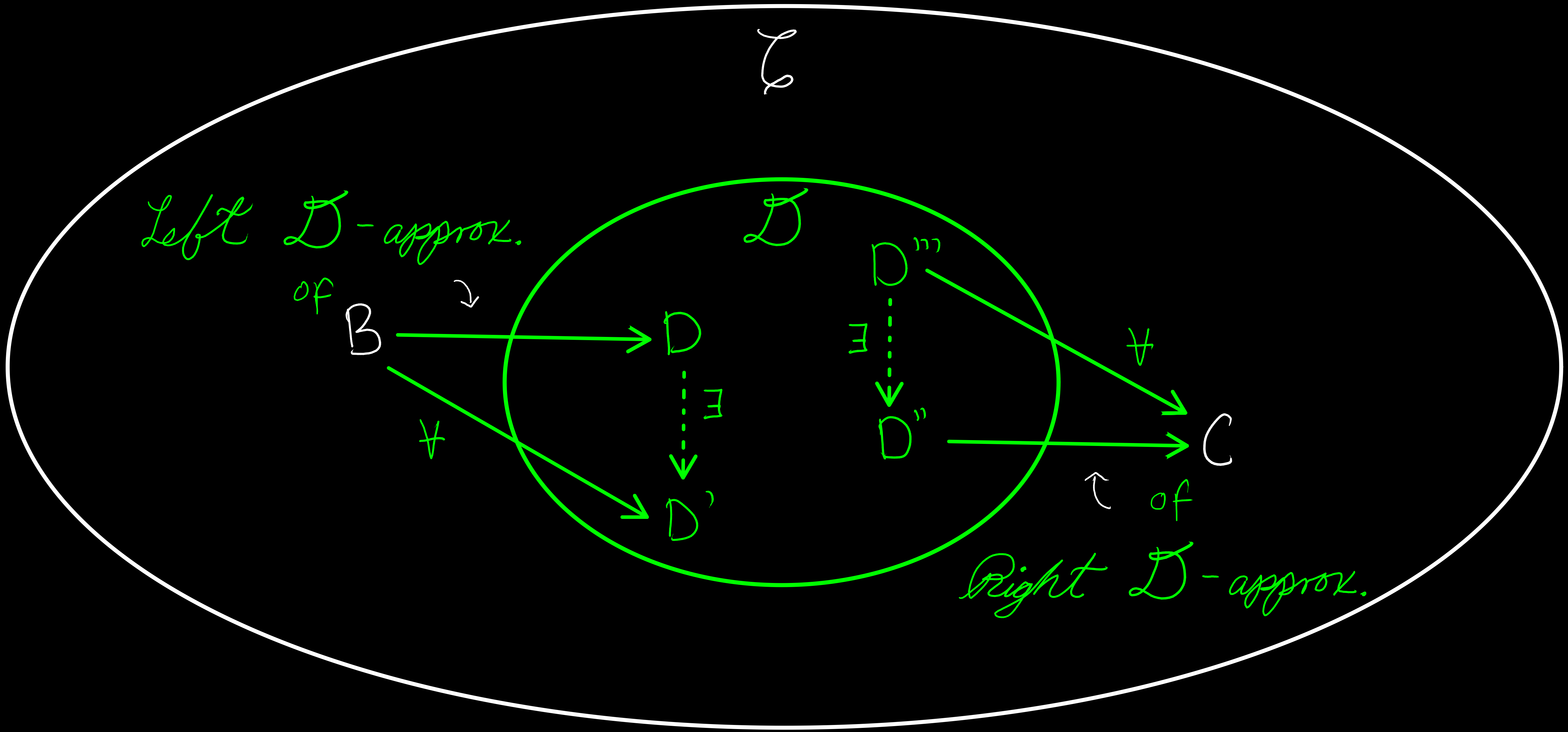
- A conflation relates certain "homological" algebraic properties between three objects

$$\begin{array}{ccccc} & \textit{inflation} & & \textit{deflation} & \\ \underset{(\text{of } g)}{\textit{cocone}} \; A & \xrightarrow{f} & B & \xrightarrow{g} & C \; \underset{(\text{of } f)}{\textit{cone}} \\ & & \underset{(\text{of } A \text{ by } C)}{\textit{extension}} & & \end{array}$$

Exp. • In mod($\mathbb{K}$): inflations = injective lin. transf.; deflations = surjective lin. transf.; cocones = kernels; cones = cokernels.

- If $V \in$ mod($\mathbb{K}$) has an inner product, $W \xrightarrow{i} V \xrightarrow{\pi} W^{\perp}$ is a conflation $\forall W \leq W$, codifying properties such as dimension.

Approximations



Silting mutation

- Let $S = \bigoplus_{i=1}^{n} S_i \in K^b(\operatorname{proj}\Lambda)$ be a silting complex, $j \in \{1, 2, 3, \ldots, n\}$, and consider

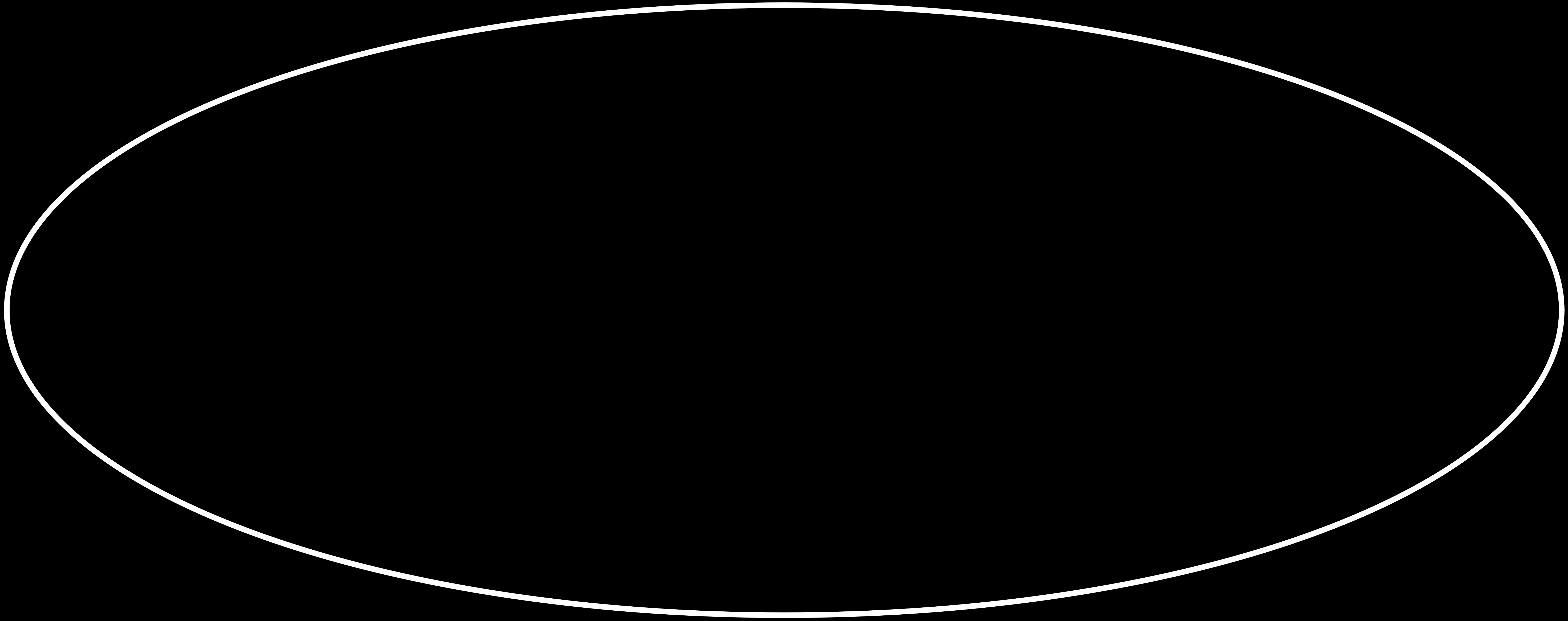
$$S_j \xrightarrow{\text{left } \operatorname{add}(D)\text{-approx.}} \bigoplus_{i=1,\, i\neq j}^{n} S_i =: D \to S_j'$$

conflation in $K^b(\operatorname{proj}\Lambda)$.

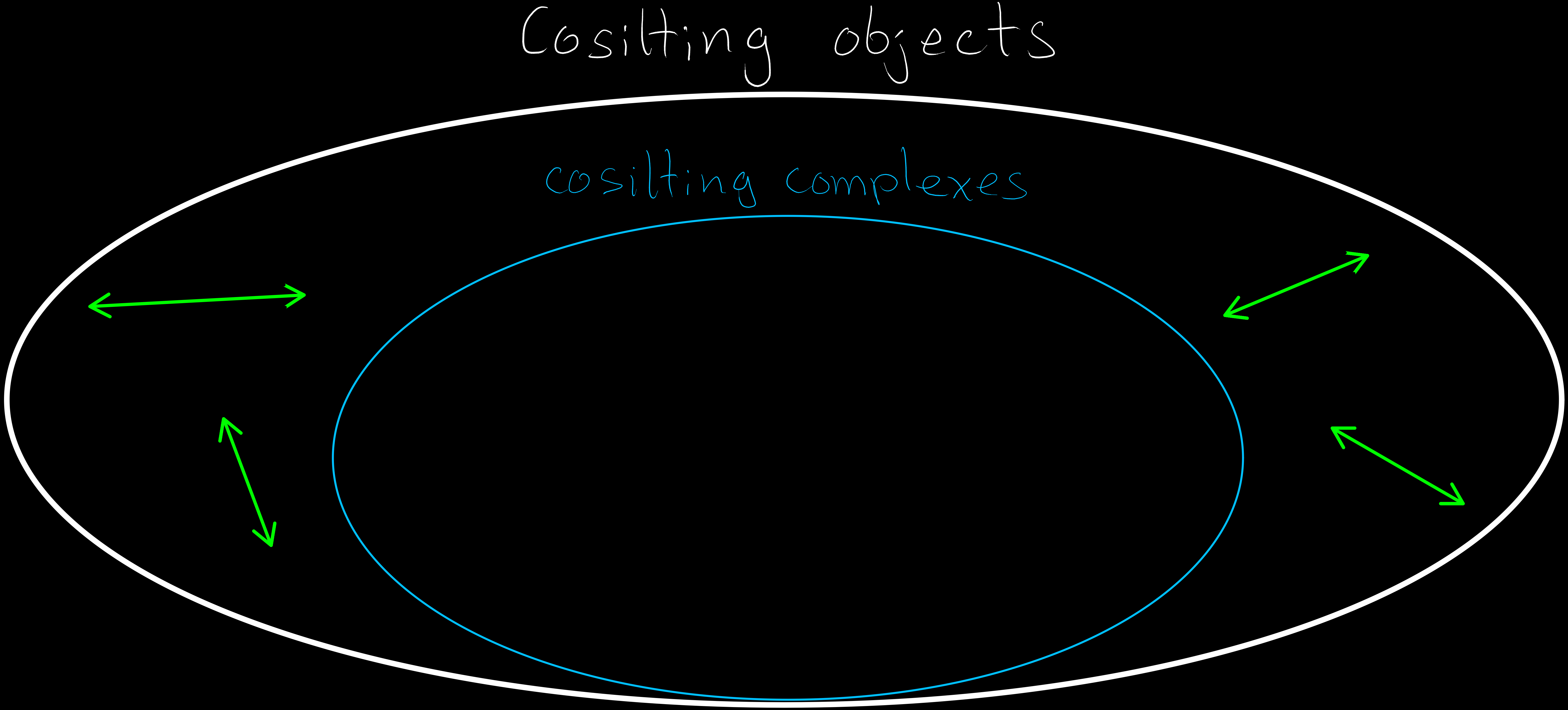
Then the *left mutation of S at j* is $D \oplus S_j'$.

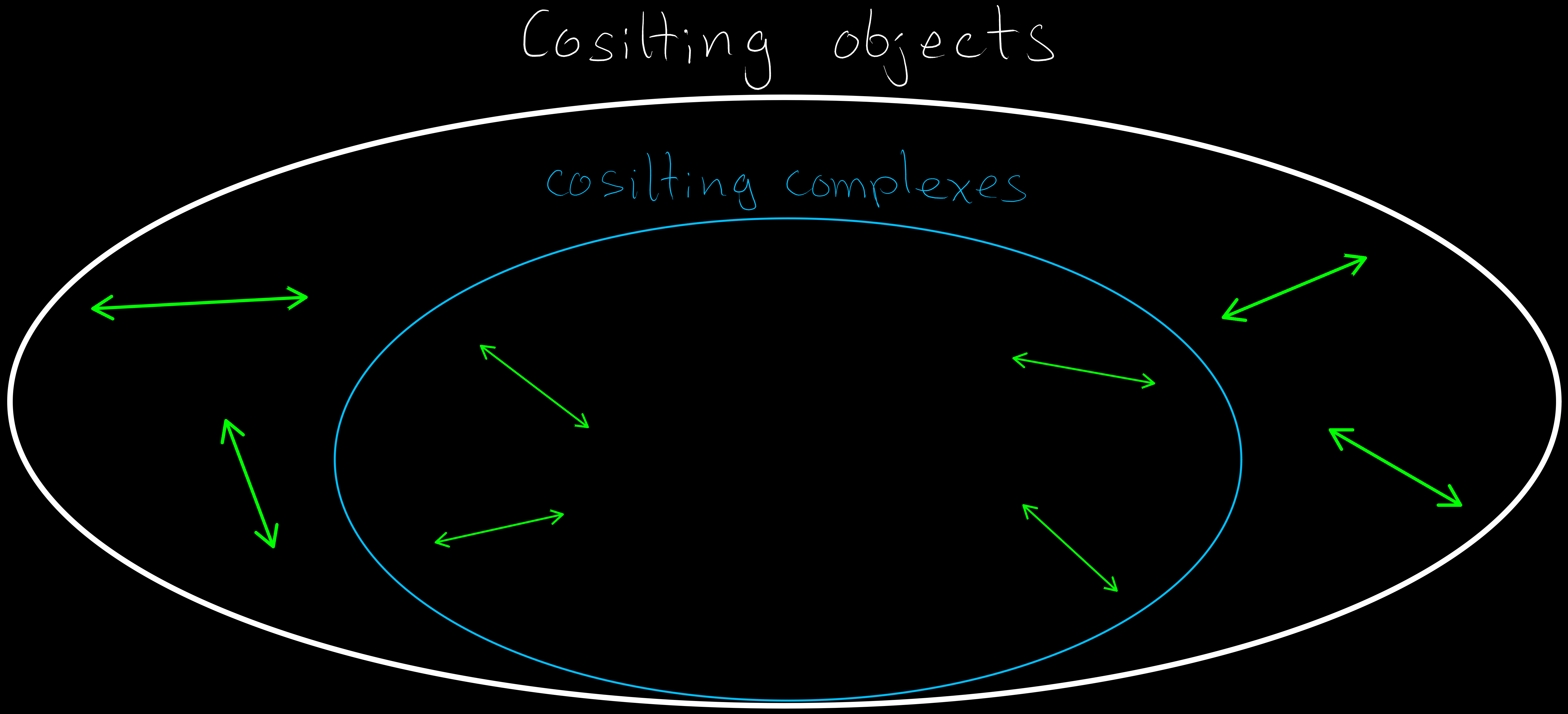
Dually, the *right mutation of $D \oplus S_j'$ at j* is $D \oplus S_j$.

Cosilting objects



Cositting objects


Cosilting objects
cosilting complexes


Cosilting objects
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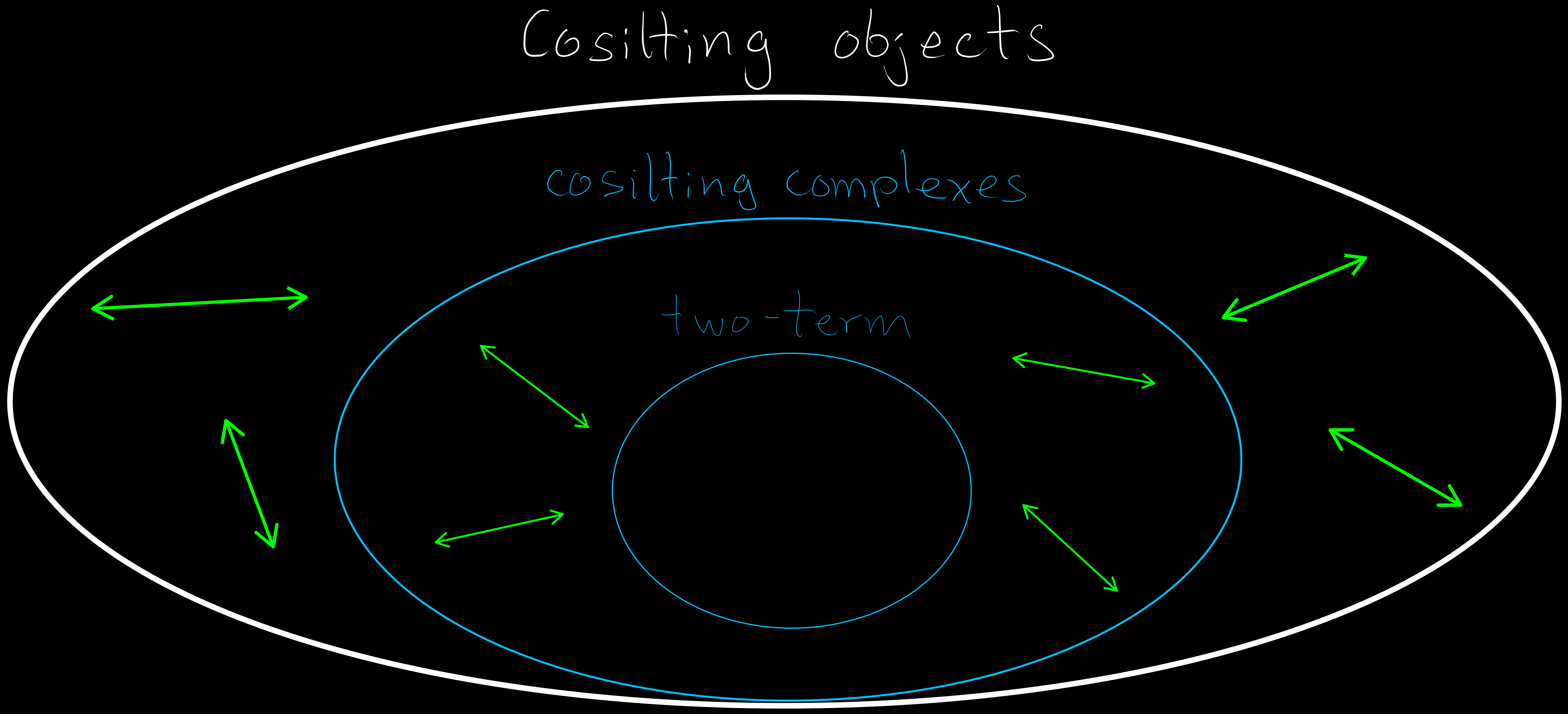
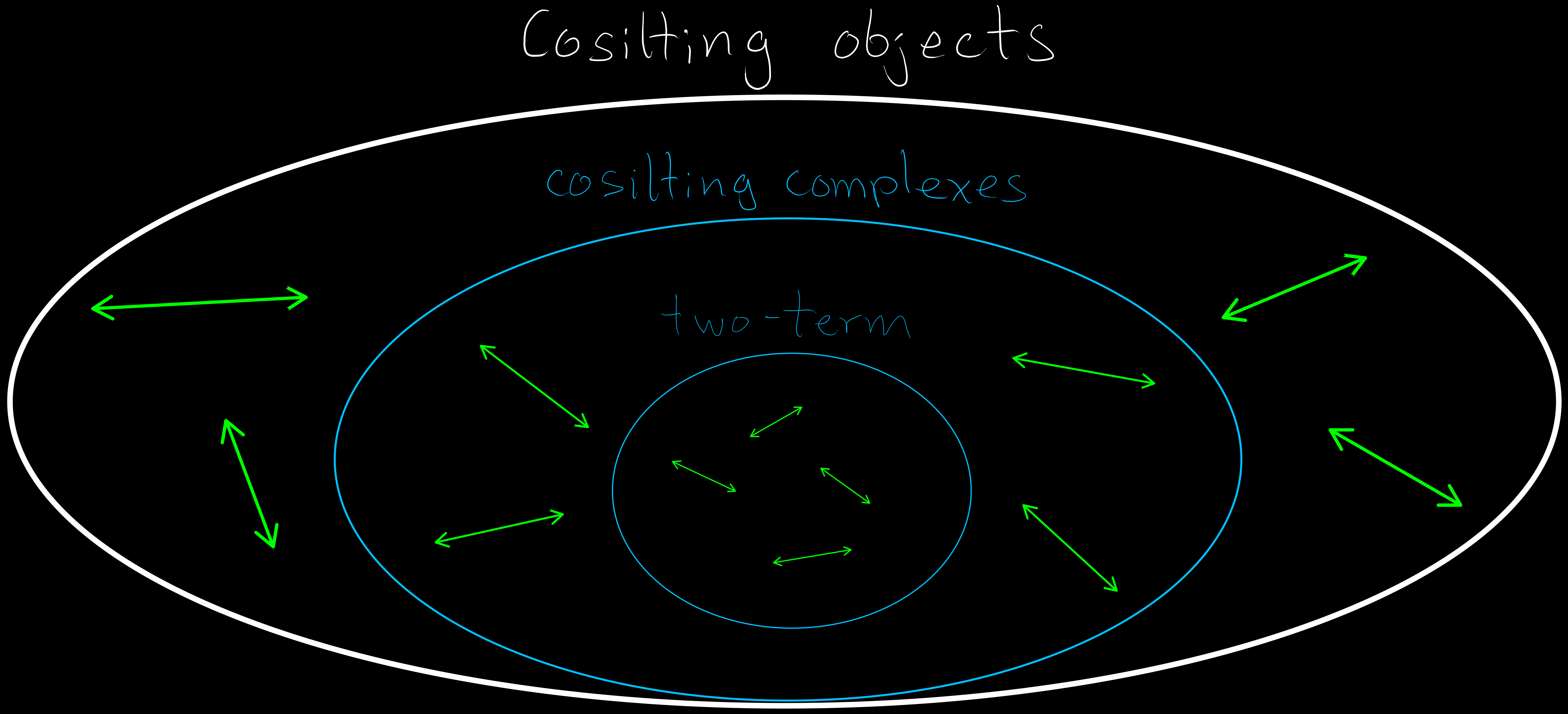
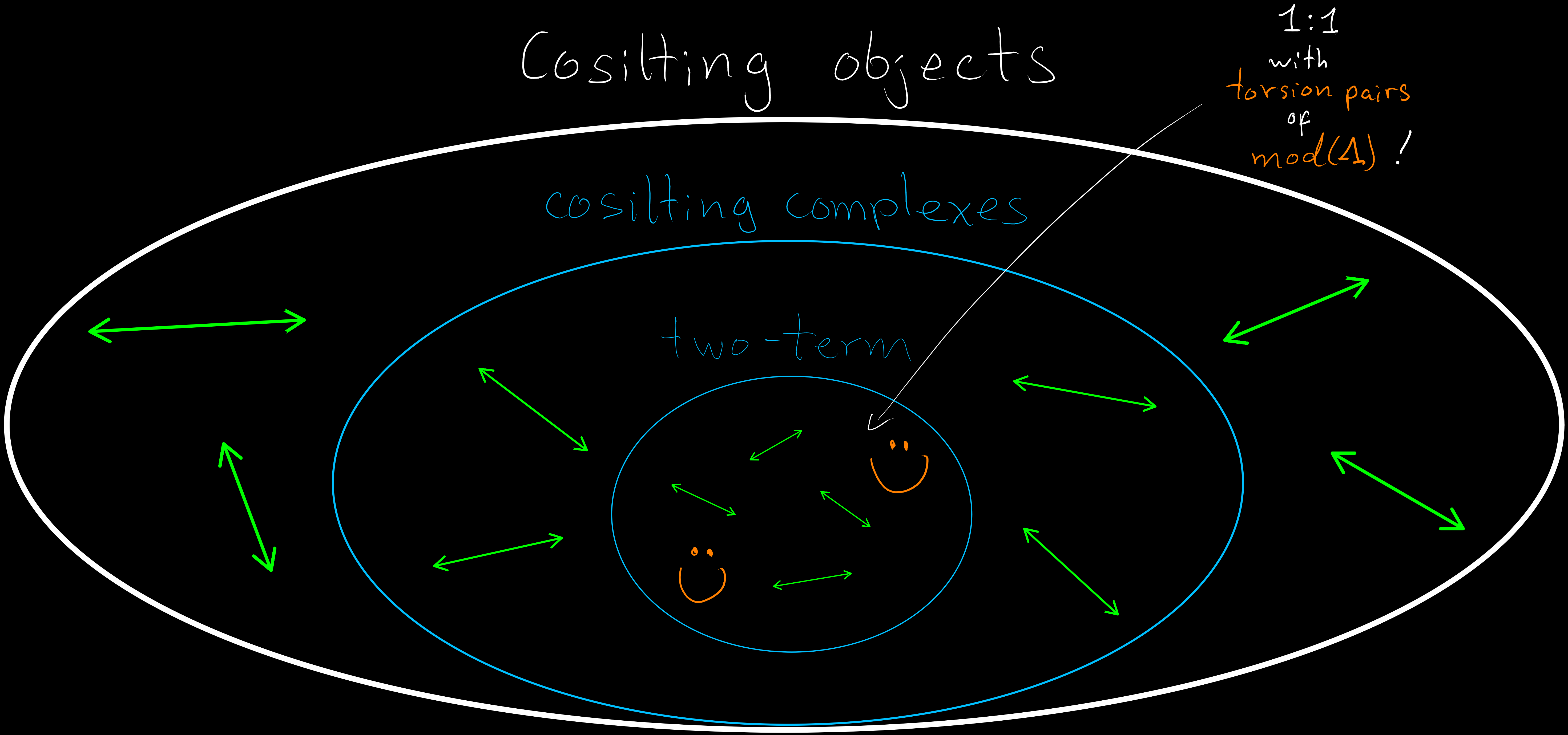
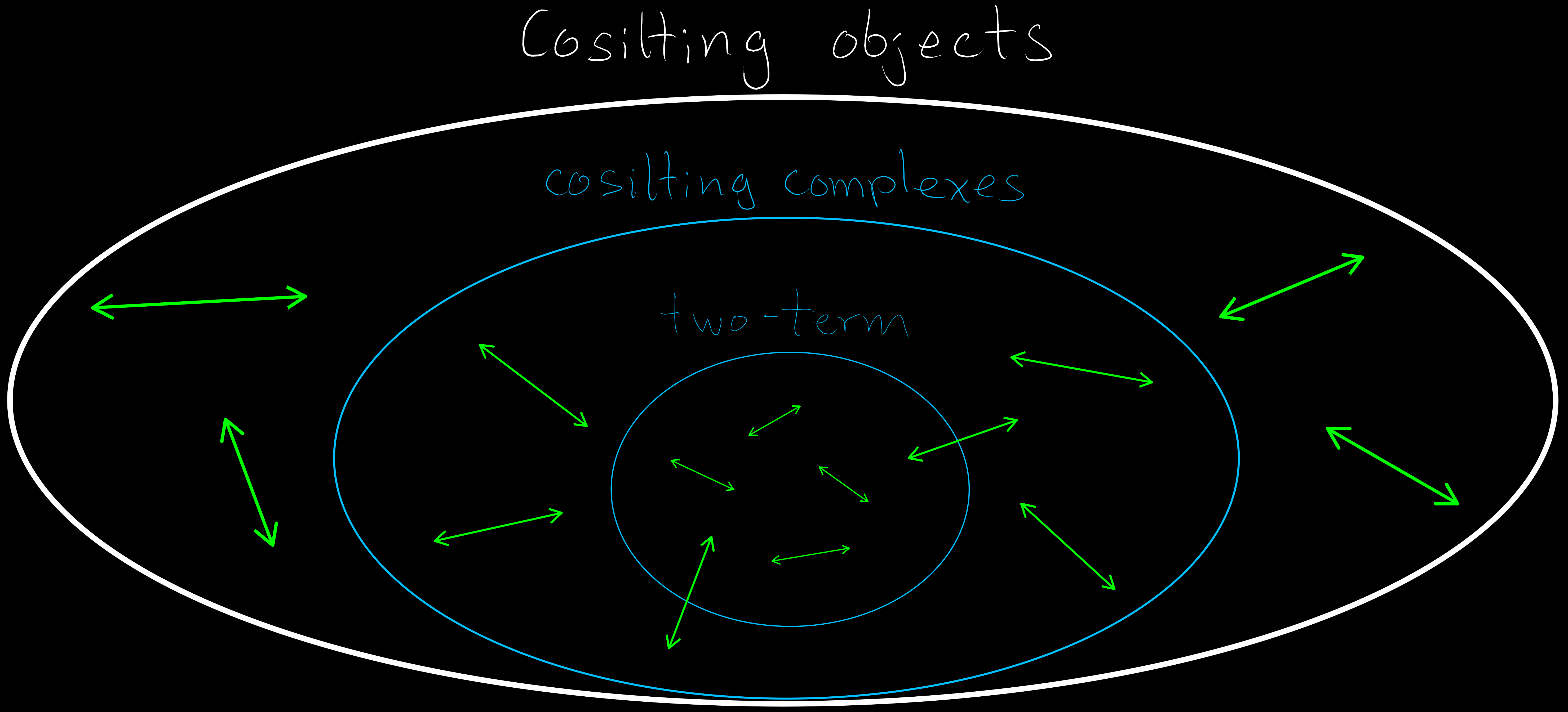

Cosilting objects
cosilting complexes
two-term


Cosilting objects
cosilting complexes
two-term

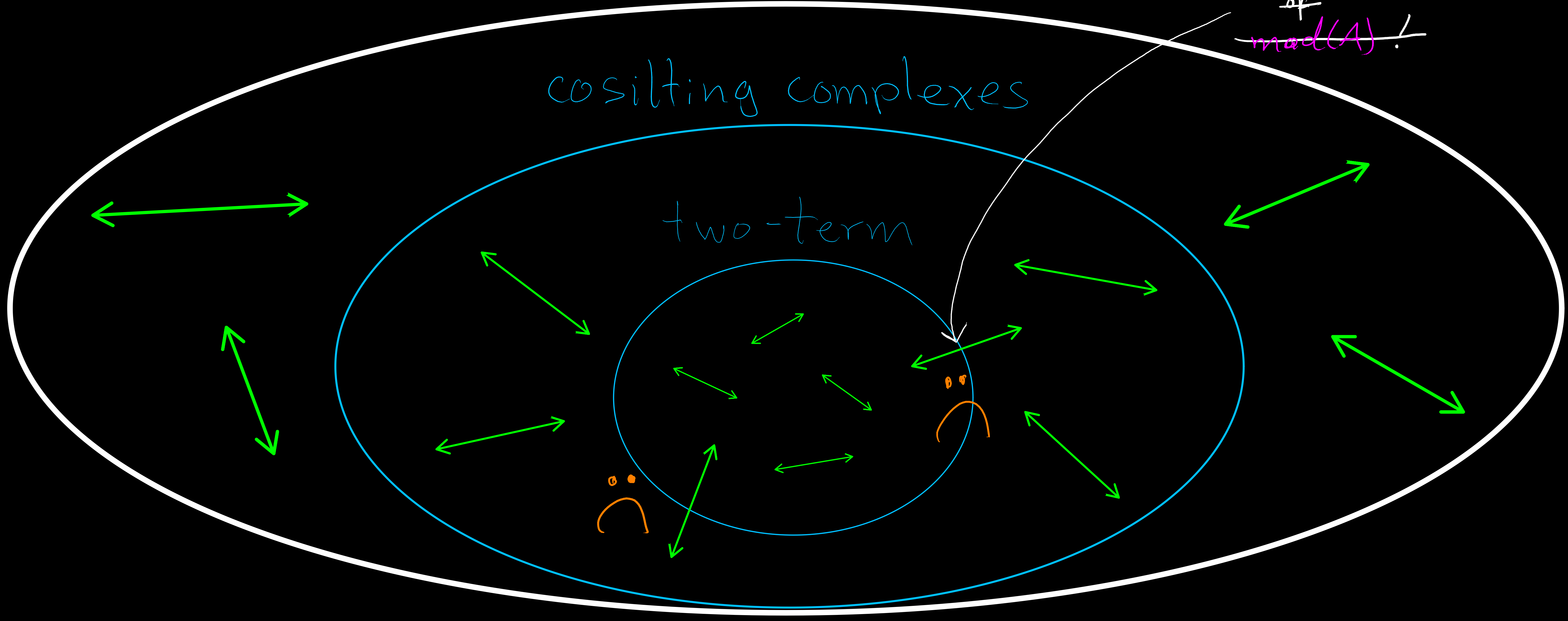

Cosilting objects
1:1
with
torsion pairs
of
mod(Λ) !
cosilting complexes
two-term



Cosilting objects
cosilting complexes
two-term

Cosilting objects

~~1:1~~
~~with~~
~~torsion pairs~~
~~of~~
~~mod(A)~~ !



Silting subcategories

- [Aihara–Iyama '12] Silting mutation can also be defined at the level of subcategories: If $\mathcal{S}$ is a silting subcat. and $\mathcal{D} \subseteq \mathcal{S}$ is such that, for each $S \in \mathcal{S}$, there exists a conflation

$$S \xrightarrow{\text{left } \mathcal{D}\text{-approx.}} D_S \to N_S,$$

then the *left mutation of $\mathcal{S}$ at $\mathcal{D}$* is $\mathrm{add}(\mathcal{D} \cup \{N_S\}_{S \in \mathcal{S}})$ (dually for right!).

- $S \in \mathbb{K}^b(\mathrm{proj}\,\Lambda)$ is a silting complex iff $\mathrm{add}(S) \subseteq \mathbb{K}^b(\mathrm{proj}\,\Lambda)$ is a silting subcategory.
- [García '23] $S \in \mathbb{K}^{[-1,0]}(\mathrm{proj}\,\Lambda)$ is a 2-term silt. cpx. iff $\mathrm{add}(S) \subseteq \mathbb{K}^{[-1,0]}(\mathrm{proj}\,\Lambda)$ is a silting subcategory.

3. Copresentations

Cosilting complexes

Dfn. A complex $\omega \in D(\mathrm{Mod}(\Lambda))$ is *cosilting* if $\mathrm{Hom}_{D(\mathrm{Mod}(\Lambda))}(\omega^J, \omega[i]) = 0$ for any set J and $i > 0$ and the closure of ω under products, summands, extensions, cones and cocones $\mathrm{thick}(\mathrm{Prod}(\omega))$ is $K^b(\mathrm{Inj}\,\Lambda)$. Moreover, it is *2-term* if $\omega \in K^{[0,1]}(\mathrm{Inj}\,\Lambda)$. $\rightsquigarrow$ cat. of large injective copresentations

- $\omega \in D(\mathrm{Mod}(\Lambda))$ is a cosilt. cpx. iff $\mathrm{Prod}(\omega)$ is a silt. subcat. of $K^b(\mathrm{Inj}\,\Lambda)$.
- [B'26] $\omega \in D(\mathrm{Mod}(\Lambda))$ is a 2-t.c.c. iff $\mathrm{Prod}(\omega)$ is a silt. subcat. of $K^{[0,1]}(\mathrm{Inj}\,\Lambda)$. (with $K^{[0,1]}(\mathrm{Inj}\,\Lambda) \subseteq K^b(\mathrm{Inj}\,\Lambda)$)
- [B'26] If $S \subseteq K^{[0,1]}(\mathrm{Inj}\,\Lambda)$ is product-closed silt., $\exists\, \omega \in D(\mathrm{Mod}(\Lambda))$ s.t. $S = \mathrm{Prod}(\omega)$.

Qst. Does there exist a mutation for product-closed silting subcategories of $K^{[0,1]}(\mathrm{Inj}\,\Lambda)$? How does it relate to cosilting mutation?

Large silting subcategories

• [B'26] Large silting mutation can also be defined at the level of subcategories:
If $\mathcal{S}$ is a ~~silting~~ large silting subcat. (i.e., product-closed) and $\mathrm{Prod}(D) \subseteq \mathcal{S}$ is such that, for each $S \in \mathcal{S}$, there exists a conflation

$$S \xrightarrow{\text{left } \mathrm{Prod}(D)\text{-approx.}} D_S \to N_S,$$

then the *left mutation of $\mathcal{S}$ at $\mathrm{Prod}(D)$* is $\mathrm{Prod}(D \cup \{N_S\}_{S \in \mathcal{S}})$ (dually for right!).

• [B'26] $S \in \mathbb{K}^b(\mathrm{Inj}\,R)$ (R a ring) is a <u>cosilting complex</u> iff $\mathrm{Prod}(S) \subseteq \mathbb{K}^b(\mathrm{Inj}\,R)$ is a <u>silting subcategory</u>.

• [B'26] $S \in \mathbb{K}^{[0,1]}(\mathrm{Inj}\,R)$ is a <u>2-term cosilt. cpx.</u> iff $\mathrm{Prod}(S) \subseteq \mathbb{K}^{[0,1]}(\mathrm{Inj}\,R)$ is a <u>silting subcategory</u>.

<u>Qst.</u> What else?

4. Abstractions

Abstract large silting mutation

Stp. Let $\mathcal{C}$ be extriangulated w/ positive extensions, exact products and an injective cogen.

Exp. • If $m \in \mathbb{Z}_{>0}$ and $\mathcal{T}$ is Δ'd w/ Π's & cosilt. obj. Q, take $\mathcal{C} := \mathrm{Prod}(Q)[-m] * \mathrm{Prod}(Q)[-m+1] * \cdots * \mathrm{Prod}(Q)$.

• If $m = 1$, $\mathcal{T} = D(\mathrm{Mod}\,R)$ and $Q = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}) \Rightarrow \mathcal{C} = \mathrm{Prod}(Q)[-1] * \mathrm{Prod}(Q) \cong K^{[0,1]}(\mathrm{Inj}\,R)$.

Thm. [B'26] There is a well-defined theory for mutation of *large* silting subcategories.

Thm. [B'26] There exists a bijection

$$\{C \in \mathcal{C} \mid \mathrm{Prod}(C) \subseteq_{\text{silt.}} \mathcal{C}\}/\sim$$

$$\updownarrow \; 1{:}1$$

$$\{\mathcal{S} \subseteq_{\text{silt.}} \mathcal{C} \mid \mathrm{Prod}(\mathcal{S}) = \mathcal{S} \;\&\; \mathrm{id}(S) < \infty \;\forall S \in \mathcal{S}\},$$

where $C \sim C'$ if $\mathrm{Prod}(C) = \mathrm{Prod}(C')$, connecting mutation at both levels.

Closing remarks

Rmk. • Since $\mathrm{id}(K^{[0,1]}(\mathrm{Inj}R)) \leq 1$, these results give a theory of mutation for 2-term cosilting complexes (thus, torsion pairs in $\mathrm{mod}(\Lambda)$) via large silting mutation in $K^{[0,1]}(\mathrm{Inj}R)$.

• If $\mathrm{id}(R) < m \rightsquigarrow$ mutation of m-cotilting modules via large silt. mutation.

• The dual theory (for coproducts) also holds!

Qst. Large mutability criteria?

Qst. Can large silting mutations be given geometric models? That is, can they be used to categorify the mutations of "large" cluster-tilting objects?

Grazie!