Mutation in
0-Auslander
Extriangulated Categories

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Mutation in 0-Auslander Extriangulated Categories

1. Unifying homological framework: extriangulated categories

mutation

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§1. Unifying homological framework:

extriangulated categories

§1.1 Partial actions on $\mathbb{E}$-extensions

Stp. Let $\mathscr{C}$ be an additive category and $\mathbb{E}:\mathscr{C}^{op}\times\mathscr{C}\to Ab$ be an additive bifunctor.

Rmk. (1) The correspondence

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(A,A')\times\mathbb{E}(C,A)&\to\mathbb{E}(C,A'),\\ (a,\delta)&\mapsto a\cdot\delta:=\mathbb{E}(C,a)(\delta),\end{aligned}$$

when defined for all $A,A',C\in\mathscr{C}$, satisfies the following properties:

(i) $1_A\cdot\delta=\delta\quad\forall\,\delta\in\mathbb{E}(C,A)$;

(ii) $(a'a)\cdot\delta=a'\cdot(a\cdot\delta)\quad\forall\,a\in\mathrm{Hom}_{\mathscr{C}}(A,A'),a'\in\mathrm{Hom}_{\mathscr{C}}(A',A''),\delta\in\mathbb{E}(C,A)$;

(iii) $a\cdot(\delta_1+\delta_2)=a\cdot\delta_1+a\cdot\delta_2\quad\forall\,a\in\mathrm{Hom}_{\mathscr{C}}(A,A'),\delta_1,\delta_2\in\mathbb{E}(C,A)$;

(iv) $(a_1+a_2)\cdot\delta=a_1\cdot\delta+a_2\cdot\delta\quad\forall\,a_1,a_2\in\mathrm{Hom}_{\mathscr{C}}(A,A'),\delta\in\mathbb{E}(C,A)$;

(v) $0\cdot\delta=0\quad\forall\,\delta\in\mathbb{E}(C,A)$;

That is, it defines a partial left action compatible with the Hom-groups' and $\mathbb{E}$-groups' sum operations.

(2) Dually, the correspondence

$$\mathbb{E}(C,A) \times \mathrm{Hom}_{\mathscr{C}}(C',C) \to \mathbb{E}(C',A),$$
$$(\delta, c) \mapsto \delta \cdot c := \mathbb{E}(c^{\mathrm{op}}, A)(\delta),$$

when defined for all $A, C, C' \in \mathscr{C}$, satisfies the following properties:

(vi) $\delta \cdot 1_C = \delta \quad \forall\, \delta \in \mathbb{E}(C,A)$;

(vii) $\delta \cdot (cc') = (\delta \cdot c) \cdot c' \quad \forall\, c \in \mathrm{Hom}_{\mathscr{C}}(C',C), c' \in \mathrm{Hom}_{\mathscr{C}}(C'',C'), \delta \in \mathbb{E}(C,A)$;

(viii) $(\delta_1 + \delta_2) \cdot c = \delta_1 \cdot c + \delta_2 \cdot c \quad \forall\, c \in \mathrm{Hom}_{\mathscr{C}}(C',C), \delta_1, \delta_2 \in \mathbb{E}(C,A)$;

(ix) $\delta \cdot (c_1 + c_2) = \delta \cdot c_1 + \delta \cdot c_2 \quad \forall\, c_1, c_2 \in \mathrm{Hom}_{\mathscr{C}}(C',C), \delta \in \mathbb{E}(C,A)$;

(x) $\delta \cdot 0 = 0 \quad \forall\, \delta \in \mathbb{E}(C,A)$.

That is, it defines a partial right action compatible with the Hom-groups' and $\mathbb{E}$-groups' sum operations.

Definition 1.2.1.2 [NP19, Definitions 2.7 & 2.8] Let $A, C \in \mathscr{C}$. Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ are *equivalent* if there exists an isomorphism $b : B \xrightarrow{\sim} B'$ such that the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, equal] & B \arrow[r, "y"] \arrow[d, "b", "\wr"'] & C \arrow[d, equal] \\ A \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C. \end{tikzcd}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$. In particular, we denote

$$0 := [A \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} A \bigoplus C \xrightarrow{(0\ 1)} C].$$

Moreover, for each pair of equivalence classes $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A'' \xrightarrow{x''} B'' \xrightarrow{y''} C'']$, we denote

$$[A \xrightarrow{x} B \xrightarrow{y} C] \bigoplus [A'' \xrightarrow{x''} B'' \xrightarrow{y''} C''] := [A \bigoplus A'' \xrightarrow{x \bigoplus x''} B \bigoplus B'' \xrightarrow{y \bigoplus y''} C \bigoplus C''].$$

Definition 1.2.1.3 [NP19, Definitions 2.9 & 2.10] An *additive realization* of $\mathbb{E}$ is a correspondence $\mathfrak{s}$ which, for all $A, C \in \mathscr{C}$, associates to each $\delta \in \mathbb{E}(C, A)$ an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ such that the following properties are satisfied:

(i) if $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ and there exist $a \in \mathrm{Hom}_{\mathscr{C}}(A, A'), c \in \mathrm{Hom}_{\mathscr{C}}(C', C)$ such that $a \cdot \delta = \delta' \cdot c$, then there exists $b \in \mathrm{Hom}_{\mathscr{C}}(B, B')$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}$$

in $\mathscr{C}$ commutes;

(ii) for all $A, C \in \mathscr{C}$ and $0 \in \mathbb{E}(C, A)$, we have that $\mathfrak{s}(0) = [A \xrightarrow{\binom{1}{0}} A \bigoplus C \xrightarrow{(0\ 1)} C]$;

(iii) if $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, then $\mathfrak{s}(\delta \bigoplus \delta') = \mathfrak{s}(\delta) \bigoplus \mathfrak{s}(\delta')$.

In this case, if $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, then $A \xrightarrow{x} B \xrightarrow{y} C$ is known as a *realization* of δ.

Stp. Let $\mathscr{C}$ and $\mathbb{E}: \mathscr{C}^{op} \times \mathscr{C} \to Ab$ be additive and $\mathfrak{s}$ be an additive realization of $\mathbb{E}$.

Dfn. For any $A, C \in \mathscr{C}$ and $\delta \in \mathbb{E}(C, A)$, we use the following terminology.

$\mathbb{E}$-extension

inflation deflation

$$\mathfrak{s}(\delta) = [\,\overbrace{A \xrightarrow{x} B \xrightarrow{y} C}^{\mathfrak{s}\text{-conflation}}\,] \equiv \overbrace{A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}}^{\mathbb{E}\text{-triangle}}$$

cocone extension cone

We may drop the prefixes $\mathbb{E}$- and $\mathfrak{s}$- when it does not lead to any confusion.

§1.3 Long exact sequences

Prp. [NP19, Prp. 3.3] The following conditions are equivalent.

(a) For any $\mathbb{E}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$ and $X \in \mathscr{C}$, we have the following exact sequences in Ab

$$\mathrm{Hom}_{\mathscr{C}}(X,A) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(X,f)} \mathrm{Hom}_{\mathscr{C}}(X,B) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(X,g)} \mathrm{Hom}_{\mathscr{C}}(X,C) \xrightarrow{\delta_\#} \mathbb{E}(X,A) \xrightarrow{\mathbb{E}(X,f)} \mathbb{E}(X,B),$$

$$\mathrm{Hom}_{\mathscr{C}}(C,X) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(g,X)} \mathrm{Hom}_{\mathscr{C}}(B,X) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(f,X)} \mathrm{Hom}_{\mathscr{C}}(A,X) \xrightarrow{\delta^\#} \mathbb{E}(C,X) \xrightarrow{\mathbb{E}(g,X)} \mathbb{E}(B,X).$$

(b) All diagrams in $\mathscr{C}$ of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d] & C \arrow[r, dashed, "\delta"] & {} \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} A \arrow[r] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r, dashed, "\delta"] \arrow[d] & {} \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd}$$

can be completed into a fully commutative diagram in $\mathscr{C}$.

$\Big\}$ (ET3) & (ET3)* or "completion of morphisms of $\mathbb{E}$-triangles"

Prp. [NP19, Prp. 3.11] If (ET3), (ET3)*, the *octahedral axiom* for $\mathbb{E}$-triangles (ET4) and its dual (ET4)* are satisfied then, for any $\mathbb{E}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$ and $X \in \mathscr{C}$, we have the following exact sequences in Ab

$$\mathrm{Hom}_{\mathscr{C}}(X,A) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(X,f)} \mathrm{Hom}_{\mathscr{C}}(X,B) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(X,g)} \mathrm{Hom}_{\mathscr{C}}(X,C) \xrightarrow{\delta_\#} \mathbb{E}(X,A) \xrightarrow{\mathbb{E}(X,f)} \mathbb{E}(X,B) \xrightarrow{\mathbb{E}(X,g)} \mathbb{E}(X,C),$$

$$\mathrm{Hom}_{\mathscr{C}}(C,X) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(g,X)} \mathrm{Hom}_{\mathscr{C}}(B,X) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(f,X)} \mathrm{Hom}_{\mathscr{C}}(A,X) \xrightarrow{\delta^\#} \mathbb{E}(C,X) \xrightarrow{\mathbb{E}(g,X)} \mathbb{E}(B,X) \xrightarrow{\mathbb{E}(f,X)} \mathbb{E}(A,X).$$

Dfn. An *extriangulated category* is a triple $(\mathscr{C},\mathbb{E},\mathfrak{s})$ such that $\mathscr{C}$ is an additive category, $\mathbb{E}$ is an additive bifunctor on $\mathscr{C}$, $\mathfrak{s}$ is an additive realization of $\mathbb{E}$ and the axioms (ET3), (ET4) and their duals are satisfied.

Rmk. Extriangulated categories are self-dual.

Exp. (1) Exact* (e.g. abelian) and triangulated (e.g. homotopy, derived) categories.

(2) Extension closed subcategories of triangulated categories (e.g. $K^{[-1,0]}(\mathrm{proj}(\Lambda)) \subseteq K^b(\mathrm{proj}(\Lambda))$).

Definition 1.2.2.1 Let $\mathcal{C}$ and $\mathcal{D}$ be classes of objects in $\mathscr{C}$.

(a) We denote by $\mathcal{C} * \mathcal{D}$ the class of objects E in $\mathscr{C}$ for which there exists a $\mathfrak{s}$-conflation $C \to E \to D$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

(b) We denote by $\mathrm{Cone}(\mathcal{C}, \mathcal{D})$ the class of objects E in $\mathscr{C}$ for which there exists a $\mathfrak{s}$-conflation $C \to D \to E$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

(c) We denote by $\mathrm{Cocone}(\mathcal{C}, \mathcal{D})$ the class of objects E in $\mathscr{C}$ for which there exists a $\mathfrak{s}$-conflation $E \to C \to D$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

Definition 1.2.2.2 A class $\mathcal{C}$ of objects in $\mathscr{C}$ is

(a) *closed under extensions* in $\mathscr{C}$ if, for each $\mathfrak{s}$-conflation $A \to B \to C$ such that $A, C \in \mathcal{C}$, it follows that $B \in \mathcal{C}$, i.e. if $\mathcal{C} * \mathcal{C} \subseteq \mathcal{C}$;

(b) *closed under cones* in $\mathscr{C}$ if, for each $\mathfrak{s}$-conflation $A \to B \to C$ such that $A, B \in \mathcal{C}$, it follows that the cone $C \in \mathcal{C}$, i.e. if $\mathrm{Cone}(\mathcal{C}, \mathcal{C}) \subseteq \mathcal{C}$;

(c) *closed under cocones* in $\mathscr{C}$ if, for each $\mathfrak{s}$-conflation $A \to B \to C$ such that $B, C \in \mathcal{C}$, it follows that the cocone $A \in \mathcal{C}$, i.e. if $\mathcal{C} \supseteq \mathrm{Cocone}(\mathcal{C}, \mathcal{C})$.

§1.4 Projectivity and injectivity

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

Dfn.

(a) An object $P \in \mathscr{C}$ is $\mathbb{E}$-*projective* if all morphisms from P to the cone of a $\mathfrak{s}$-conflation factorize through the corresponding deflation, i.e., if $A \xrightarrow{f} B \xrightarrow{g} C$ is a $\mathfrak{s}$-conflation and $P \xrightarrow{h} C$ is a morphism in $\mathscr{C}$, then there exists a morphism $P \xrightarrow{h'} B$ in $\mathscr{C}$ such that the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd} & & P \arrow[d, "h"] \arrow[dl, dotted, "h'"'] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C. \end{tikzcd}$$

(b) $\mathbb{E}$-*injective* objects are defined dually.

(c) An object $Q \in \mathscr{C}$ is $\mathbb{E}$-*projective-injective* if it is both $\mathbb{E}$-projective and $\mathbb{E}$-injective.

(d) We denote by $\mathrm{Proj}_{\mathbb{E}}(\mathscr{C})$ and $\mathrm{Inj}_{\mathbb{E}}(\mathscr{C})$ the classes of $\mathbb{E}$-projective objects and of $\mathbb{E}$-injective objects in $\mathscr{C}$, respectively.

(e) $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is *reduced* if $\mathrm{Proj}_{\mathbb{E}}(\mathscr{C}) \cap \mathrm{Inj}_{\mathbb{E}}(\mathscr{C}) = \{0\}$.

Lemma 1.2.2.13 (Characterization of $\mathbb{E}$-projectives and $\mathbb{E}$-injectives) Let $Q \in \mathscr{C}$.

(a) $Q \in \mathrm{Proj}_{\mathbb{E}}(\mathscr{C})$ if, and only if, $\mathbb{E}(Q, A) = 0$ for all $A \in \mathscr{C}$.

(b) $Q \in \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})$ if, and only if, $\mathbb{E}(B, Q) = 0$ for all $B \in \mathscr{C}$.

[NP19, Prp. 3.24]

Proof. Exercise! (Hint: Use long exact sequences).

Proposition 1.2.2.17 [NP19, Propopsition 3.30] Let $\mathscr{I} \subseteq \mathscr{C}$ be a full additive subcategory closed under isomorphisms in $\mathscr{C}$ such that $\mathscr{I} \subseteq \mathrm{Proj}_{\mathbb{E}}(\mathscr{C}) \cap \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})$, and let $\langle \mathscr{I} \rangle$ be the ideal generated by the morphisms in $\mathscr{C}$ which factorize through obects in $\mathscr{I}$. Then the quotient category $\mathscr{C}/\langle \mathscr{I} \rangle$ has an extriangulated structure, induced by that of $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$. In particular,

$$\mathscr{C}' := \mathscr{C}/\langle \mathrm{Proj}_{\mathbb{E}}(\mathscr{C}) \cap \mathrm{Inj}_{\mathbb{E}}(\mathscr{C}) \rangle$$

is a reduced extriangulated category.

§1.5 Resolutions and coresolutions

Definition 1.2.2.28 Let $\mathcal{S}$ be a class of objects in $\mathcal{C}$ and $C \in \mathscr{C}$.

(a) A $\mathcal{S}$-*resolution* of C is a sequence of $\mathfrak{s}$-conflations $L_{i+1} \to S_i \to L_i$ with $i \geq 0$ such that $L_0 = C$ and $S_i \in \mathcal{S}$ for all i. If, furthermore, there exists an integer $k > 0$ such that $L_k = 0$, and therefore a smallest such integer m, then the *length* of the $\mathcal{S}$-resolution is $m - 1$; otherwise, it is ∞.

(b) A $\mathcal{S}$-*coresolution* of C is a sequence of $\mathfrak{s}$-conflations $L_i \to S_i \to L_{i+1}$ with $i \geq 0$ such that $L_0 = C$ and $S_i \in \mathcal{S}$ for all i. If, furthermore, there exists an integer $k > 0$ such that $L_k = 0$, and therefore a smallest such integer m, then the *length* of the $\mathcal{S}$-coresolution is $m - 1$; otherwise, it is ∞.

In particular, we call $\mathrm{Proj}_{\mathbb{E}}(\mathscr{C})$-resolutions and $\mathrm{Inj}_{\mathbb{E}}(\mathscr{C})$-coresolutions *projective resolutions* and *injective coresolutions*, respectively.

Definition 1.2.2.29 Let $\mathcal{S}$ be a class of objects in $\mathscr{C}$. We recursively define the following classes:

$$\mathcal{S}_n^{\wedge} := \begin{cases} \{0\} & \text{if } n = -1, \\ \operatorname{Cone}(\mathcal{S}_{n-1}^{\wedge}, \mathcal{S}) & \text{if } n \geq 0; \end{cases}$$

$$\mathcal{S}_n^{\vee} := \begin{cases} \{0\} & \text{if } n = -1, \\ \operatorname{Cocone}(\mathcal{S}, \mathcal{S}_{n-1}^{\vee}), & \text{if } n \geq 0. \end{cases}$$

Moreover, we define

$$\mathcal{S}^{\wedge} := \bigcup_{n \geq 0} \mathcal{S}_n^{\wedge} \quad \text{and} \quad \mathcal{S}^{\vee} := \bigcup_{n \geq 0} \mathcal{S}_n^{\vee}.$$

Remark 1.2.2.30 Let $\mathcal{S} \subseteq \mathscr{C}$ and $m \geq 0$.

(1) $\mathcal{S}_m^{\wedge}$ is the class of objects of $\mathscr{C}$ which have a $\mathcal{S}$-resolution of length m; thus, $\mathcal{S}^{\wedge}$ is the class of objects of $\mathscr{C}$ which have a $\mathcal{S}$-resolution of finite length.

(2) $\mathcal{S}_m^{\vee}$ is the class of objects of $\mathscr{C}$ which have a $\mathcal{S}$-coresolution of length m; thus, $\mathcal{S}^{\vee}$ is the class of objects of $\mathscr{C}$ which have a $\mathcal{S}$-coresolution of finite length.

Definition 1.2.2.32 Let $C \in \mathscr{C}$.

(a) The *projective dimension* of C, denoted by $\operatorname{pd}(C)$, is the smallest integer $n \geq 0$ (if it exists) such that C has a projective resolution of length n.

(b) The *projective dimension* of $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is

$$\operatorname{pd}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) := \sup\{\operatorname{pd}(C) \mid C \in \mathscr{C}\}.$$

(c) The *injective dimension* of C, denoted by $\operatorname{id}(C)$, is the smallest integer $n \geq 0$ (if it exists) such that C has a injective coresolution of length n.

(d) The *injective dimension* of $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is

$$\operatorname{id}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) := \sup\{\operatorname{id}(C) \mid C \in \mathscr{C}\}.$$

Syzygies and cosyzygies

Definition 1.2.2.33 Let $\mathcal{S} \subseteq \mathscr{C}$.

(a) We recursively define the following classes:

$$\Omega^i(\mathcal{S}) := \begin{cases} \mathcal{S} & \text{if } i = 0, \\ \mathrm{Cocone}(\mathrm{Proj}_{\mathbb{E}}(\mathscr{C}), \Omega^{i-1}(\mathcal{S})) & \text{if } i \geq 1 \text{ and } \Omega^{i-1}(\mathcal{S}) \neq \varnothing, \end{cases}$$

and we call $\Omega^i(\mathcal{S})$ the *class of i-th syzygies* of $\mathcal{S}$.

(b) We recursively define the following classes:

$$\Omega^{-i}(\mathcal{S}) := \begin{cases} \mathcal{S} & \text{if } i = 0, \\ \mathrm{Cone}(\Omega^{-(i-1)}(\mathcal{S}), \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})) & \text{if } i \geq 1 \text{ and } \Omega^{-(i-1)}(\mathcal{S}) \neq \varnothing, \end{cases}$$

and we call $\Omega^{-i}(\mathcal{S})$ the *class of i-th cosyzygies* of $\mathcal{S}$.

Remark 1.2.2.34

(1) For $\mathcal{S} \subseteq \mathscr{C}$ and $j > 1$, $\Omega^1(\mathcal{S})$ may be empty and, moreover, $\Omega^j(\mathcal{S})$ is undefined if $\Omega^{j-1}(\mathcal{S}) = \varnothing$. However, in general, $\Omega^k(\mathcal{S})$ is well defined and non-empty for all $0 \leq k \leq \mathrm{pd}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) + 1$ and $\mathcal{S} \subseteq \mathscr{C}$.

(2) Dually, for $\mathcal{S} \subseteq \mathscr{C}$ and $j > 1$, $\Omega^{-1}(\mathcal{S})$ may be empty and, moreover, $\Omega^{-j}(\mathcal{S})$ is undefined if $\Omega^{-(j-1)}(\mathcal{S}) = \varnothing$. However, in general, $\Omega^{-k}(\mathcal{S})$ is well defined and non-empty for all $0 \leq k \leq \mathrm{id}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) + 1$ and $\mathcal{S} \subseteq \mathscr{C}$.

Definition 1.2.2.38 The extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has:

(a) *enough* $\mathbb{E}$-*projectives* if, for all $X \in \mathscr{C}$, there exists a deflation $P \twoheadrightarrow X$ with P an $\mathbb{E}$-projective, i.e. if $\boxed{\mathrm{Cone}(\mathscr{C}, \mathrm{Proj}_{\mathbb{E}}(\mathscr{C})) = \mathscr{C};}$ $\Rightarrow$ all objects in $\mathscr{C}$ have projective resolutions!

(b) *enough* $\mathbb{E}$-*injectives* if, for all $X \in \mathscr{C}$, there exists an inflation $X \rightarrowtail I$ with I an $\mathbb{E}$-injective, i.e. if $\boxed{\mathscr{C} = \mathrm{Cocone}(\mathrm{Inj}_{\mathbb{E}}(\mathscr{C}), \mathscr{C}).}$ $\Rightarrow$ all objects in $\mathscr{C}$ have injective coresolutions!

§1.7 Enough $\mathbb{E}$-projectives and $\mathbb{E}$-injectives

Setup 1.2.5 Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with both enough $\mathbb{E}$-projectives and $\mathbb{E}$-injectives.

Note For $C \in \mathscr{C}$ and $i \geq 1$, we will denote any i-th syzygy and i-th cosyzygy of C by $\Omega^i(C)$ and $\Omega^{-i}(C)$, respectively.

Lemma 1.2.5.1 Let $C, D \in \mathscr{C}$. Then,

$$\mathbb{E}(\Omega^1(C), D) \simeq \mathbb{E}(C, \Omega^{-1}(D)).$$

Proof. [LN17, Lmm. 5.1]

Theorem 1.2.5.2 For any $\mathfrak{s}$-conflation $A \to B \to C$ and $X \in \mathscr{C}$, we have long exact sequences

$$\begin{gathered}\operatorname{Hom}(X,A) \to \operatorname{Hom}(X,B) \to \operatorname{Hom}(X,C) \to \\ \mathbb{E}(X,A) \to \mathbb{E}(X,B) \to \mathbb{E}(X,C) \to \\ \mathbb{E}(\Omega^1(X),A) \to \mathbb{E}(\Omega^1(X),B) \to \mathbb{E}(\Omega^1(X),C) \to \\ \mathbb{E}(\Omega^2(X),A) \to \mathbb{E}(\Omega^2(X),B) \to \mathbb{E}(\Omega^2(X),C) \to \dots,\end{gathered}$$

$$\begin{gathered}\operatorname{Hom}(C,X) \to \operatorname{Hom}(B,X) \to \operatorname{Hom}(A,X) \to \\ \mathbb{E}(C,X) \to \mathbb{E}(B,X) \to \mathbb{E}(A,X) \to \\ \mathbb{E}(C,\Omega^{-1}(X)) \to \mathbb{E}(B,\Omega^{-1}(X)) \to \mathbb{E}(A,\Omega^{-1}(X)) \to \\ \mathbb{E}(C,\Omega^{-2}(X)) \to \mathbb{E}(B,\Omega^{-2}(X)) \to \mathbb{E}(A,\Omega^{-2}(X)) \to \dots.\end{gathered}$$

Proof. [LN17, Prp. 5.2].

Definition 1.2.5.3 For all $C, D \in \mathscr{C}$, we recursively define

$$\mathbb{E}^i(C, D) := \begin{cases} \operatorname{Hom}(C, D) & \text{if } i = 0, \\ \mathbb{E}(C, D) & \text{if } i = 1, \\ \mathbb{E}(\Omega^{i-1}(C), D) \simeq \mathbb{E}(C, \Omega^{-(i-1)}(D)) & \text{if } i > 1. \end{cases}$$

Rmk. We thus obtain "well defined" positive extensions $\mathbb{E}^i$, $i>1$, for $\mathbb{E}$ in the sense that they allow us to further extend our $\mathfrak{s}$-conflations' long exact sequences.

Remark 1.2.5.4 We have that

$$\operatorname{pd}(C) = \min\{n \in \mathbb{Z}_{\geq 0} \mid \mathbb{E}^j(C, X) = 0 \quad \forall\, j > n, X \in \mathscr{C}\};$$
$$\operatorname{id}(C) = \min\{n \in \mathbb{Z}_{\geq 0} \mid \mathbb{E}^j(X, C) = 0 \quad \forall\, j > n, X \in \mathscr{C}\};$$

from which it follows that $\operatorname{pd}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) = \operatorname{id}(\mathscr{C}, \mathbb{E}, \mathfrak{s})$.

Setup 1.2.4 Let $(\mathscr{C}, \mathbb{E} =: \mathbb{E}^1, \mathfrak{s})$ be an extriangulated category such that $\mathbb{E}^i$ is well defined for all $i > 1$, i.e. such that, for any $\mathfrak{s}$-conflation $A \to B \to C$ and $X \in \mathscr{C}$, we have long exact sequences

$$\begin{aligned} \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \to \\ \mathbb{E}(X, A) \to \mathbb{E}(X, B) \to \mathbb{E}(X, C) \to \\ \mathbb{E}^2(X, A) \to \mathbb{E}^2(X, B) \to \mathbb{E}^2(X, C) \to \\ \mathbb{E}^3(X, A) \to \mathbb{E}^3(X, B) \to \mathbb{E}^3(X, C) \to \dots, \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}(C, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X) \to \\ \mathbb{E}(C, X) \to \mathbb{E}(B, X) \to \mathbb{E}(A, X) \to \\ \mathbb{E}^2(C, X) \to \mathbb{E}^2(B, X) \to \mathbb{E}^2(A, X) \to \\ \mathbb{E}^3(C, X) \to \mathbb{E}^3(B, X) \to \mathbb{E}^3(A, X) \to \dots. \end{aligned}$$

Definition 1.2.4.4 For a class of objects $\mathcal{S} \subseteq \mathscr{C}$, its *right* and *left orthogonal classes* are respectively defined as

$$\mathcal{S}^{\perp_{>0}} := \{B \in \mathscr{C} \mid \mathbb{E}^i(S, B) = 0 \quad \forall\, i > 0, S \in \mathcal{S}\},$$
$${}^{\perp_{>0}}\mathcal{S} := \{A \in \mathscr{C} \mid \mathbb{E}^i(A, S) = 0 \quad \forall\, i > 0, S \in \mathcal{S}\}.$$

Lemma 1.2.4.6 (Dimension shifting) Let $\mathcal{S} \subseteq \mathscr{C}$ and $C \in \mathscr{C}$.

(a) Suppose $\{L_{i+1} \to S_i \to L_i\}_{i\geq 0}$ is a $\mathcal{S}$-resolution of C. Then, for all $D \in \mathcal{S}^{\perp_{>0}}$, $n \geq 0$ and $k > 0$,

$$\mathbb{E}^{n+k}(C, D) \simeq \mathbb{E}^k(L_n, D).$$

(b) Suppose $\{L_j \to S_j \to L_{j+1}\}_{j\geq 0}$ is a $\mathcal{S}$-coresolution of C. Then, for all $B \in {}^{\perp_{>0}}\mathcal{S}$, $n \geq 0$ and $k > 0$,

$$\mathbb{E}^{n+k}(B, C) \simeq \mathbb{E}^k(B, L_n).$$

Proof. Exercise! (Hint: Use extended long exact sequences.)

Definition 1.2.4.10 An extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is *hereditary* if $\mathbb{E}^2 = 0$, i.e. if $\mathbb{E}^2(X, Y) = 0$ for all $X, Y \in \mathscr{C}$.

Proposition 1.2.4.11 (Characterization of hereditary extriangulated categories)

(1) $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is hereditary.

(2) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has enough $\mathbb{E}$-projectives, the previous conditions are equivalent to $\mathrm{pd}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) \leq 1$.

(3) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has enough $\mathbb{E}$-injectives, the previous conditions are equivalent to $\mathrm{id}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) \leq 1$.

Proof. See [GNP23, Proposition 2.1]. □

Remark 1.2.4.12 [GNP21, Remark 3.2] If $\mathbb{E}^i = 0$ for some $i > 0$, then $\mathbb{E}^k = 0$ for all $k \geq i$.

mutation

§2. Unifying homological framework:

extriangulated categories

0-Auslander

In the following, assume that all subcategories are full, additive closed under isomorphisms

§2.1 Notions of rigidity

Dfn. A subcategory $\mathcal{R} \subseteq \mathscr{C}$ is *rigid* if $\mathbb{E}(\mathcal{R}, \mathcal{R}) = 0$. Furthermore, it is

- *maximal rigid* if, $\forall X \in \mathscr{C}$, $\mathbb{E}(\mathcal{R} \oplus \operatorname{add}(X), \mathcal{R} \oplus \operatorname{add}(X)) = 0 \Rightarrow X \in \mathcal{R}$;
- *$\mathbb{E}$-tilting* if, $\forall X \in \mathscr{C}$, $\mathbb{E}(X, \mathcal{R}) = 0 = \mathbb{E}(\mathcal{R}, X) \Rightarrow X \in \mathcal{R}$.

Moreover, an object $R \in \mathscr{C}$ is rigid/maximal rigid/$\mathbb{E}$-tilting if the subcategory $\operatorname{add}(R)$ is.

Proposition 1.2.2.25 Let $\mathcal{R} \subseteq \mathscr{C}$ be a $\mathbb{E}$-tilting subcategory. Then, $\mathcal{R}$ is maximal rigid.

Proof. Let $X \in \mathscr{C}$ be such that $\mathbb{E}(\mathcal{R} \oplus \operatorname{add}(X), \mathcal{R} \oplus \operatorname{add}(X)) = 0$. Then, in particular,

$$\begin{aligned} 0 &= \mathbb{E}(\mathcal{R} \bigoplus X, \mathcal{R} \bigoplus X) \\ &= \mathbb{E}(\mathcal{R}, \mathcal{R}) \bigoplus \mathbb{E}(X, \mathcal{R}) \bigoplus (\mathcal{R}, X) \bigoplus \mathbb{E}(X, X), \end{aligned}$$

from which it follows that $\mathbb{E}(X, \mathcal{R}) = 0 = \mathbb{E}(\mathcal{R}, X)$, which implies that $X \in \mathcal{R}$ by hypothesis. $\square$

Setup 1.2.4 Let $(\mathscr{C}, \mathbb{E} =: \mathbb{E}^1, \mathfrak{s})$ be an extriangulated category such that $\mathbb{E}^i$ is well defined for all $i > 1$.

Dfn. Let $\mathcal{S} \subseteq \mathscr{C}$. We denote by $\operatorname{thick}_{\mathfrak{s}}(\mathcal{S}) \subseteq \mathscr{C}$ the closure of $\mathcal{S}$ under extensions, cones, cocones and direct summands in $\mathscr{C}$.

Dfn. A subcategory $\mathcal{R} \subseteq \mathscr{C}$ is *self orthogonal* if $\mathbb{E}^i(\mathcal{R}, \mathcal{R}) = 0 \ \forall i > 0$. Moreover, it is *presilting* if it is closed under direct summands in $\mathscr{C}$. Furthermore, it is

- *silting* if $\operatorname{thick}_{\mathfrak{s}}(\mathcal{R}) = \mathscr{C}$;
- *n-tilting* if $\mathcal{R} \subseteq \operatorname{Proj}_{\mathbb{E}}(\mathscr{C})^{\wedge}_n$ and $\operatorname{Proj}_{\mathbb{E}}(\mathscr{C}) \subseteq \mathcal{R}^{\vee}_n$;
- *n-cotilting* if $\mathcal{R} \subseteq \operatorname{Inj}_{\mathbb{E}}(\mathscr{C})^{\vee}_n$ and $\operatorname{Inj}_{\mathbb{E}}(\mathscr{C}) \subseteq \mathcal{R}^{\wedge}_n$.

An object $R \in \mathscr{C}$ is (pre)silting/n-tilting/n-cotilting if the subcategory $\operatorname{add}(R)$ is.

§2.2 Equivalence between silting and tilting

<u>Rmk</u>. Let $\mathcal{R} \subseteq \mathcal{C}$ be a presilting subcategory. Taking inspiration from [AT22]

Let $\mathsf{proj}\mathcal{C}$ denote the subcategory of $\mathcal{C}$ consisting of all projective objects in $\mathcal{C}$. → extriangulated

Proposition 5.5. *The subcategory $\mathsf{proj}\mathcal{C}$ is a silting subcategory of $\mathcal{C}$ if and only if $\mathcal{C} = (\mathsf{proj}\mathcal{C})^{\wedge}$. In this case, a subcategory $\mathcal{T}$ of $\mathcal{C}$ is a silting subcategory if and only if $\mathcal{T}$ satisfies the following conditions.*

(1) *$\mathcal{T}$ is closed under direct summands.*
(2) $\mathcal{T} \subseteq (\mathsf{proj}\mathcal{C})^{\wedge}$.
(3) *$\mathcal{T}$ is self-orthogonal.*
(4) $\mathsf{proj}\mathcal{C} \subseteq \mathcal{T}^{\vee}$.

we make the following observations:

(1) If $\mathrm{Proj}_{\mathbb{E}}(\mathcal{C})^{\wedge} = \mathcal{C}$, then $\mathcal{R}$ being silting is equivalent to $\mathrm{Proj}_{\mathbb{E}}(\mathcal{C}) \subseteq \mathcal{R}^{\vee}$.

(2) If $\mathrm{Proj}_{\mathbb{E}}(\mathcal{C})^{\wedge}_{n} = \mathcal{C}$, then $\mathcal{R}$ being n-tilting is equivalent to $\mathrm{Proj}_{\mathbb{E}}(\mathcal{C}) \subseteq \mathcal{R}^{\vee}_{n}$.

(3) $\mathrm{Proj}_{\mathbb{E}}(\mathcal{C})^{\wedge}_{n} \subseteq \mathrm{Proj}_{\mathbb{E}}(\mathcal{C})^{\wedge}$.

Qst. Is there an n for which $\mathrm{Proj}_{\mathbb{E}}(\mathscr{C})^{\wedge}_n = \mathscr{C}$ implies that $\mathcal{R}^{\vee} = \mathcal{R}^{\vee}_n$?

Since $\mathrm{Proj}_{\mathbb{E}}(\mathscr{C})^{\wedge}_0 = \mathrm{Proj}_{\mathbb{E}}(\mathscr{C})$, we are not interested in the case $n=0$, as it implies $\mathscr{C} = \mathrm{Proj}_{\mathbb{E}}(\mathscr{C})$. On the other hand, for $n=1$, the condition $\mathrm{Proj}_{\mathbb{E}}(\mathscr{C})^{\wedge}_1 = \mathrm{Cone}(\mathrm{Proj}_{\mathbb{E}}(\mathscr{C}), \mathrm{Proj}_{\mathbb{E}}(\mathscr{C})) = \mathscr{C}$ is equivalent to $\mathscr{C}$ having enough $\mathbb{E}$-projectives and $\mathrm{pd}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) \leq 1$, which by [GNP23, Prp. 2.1] is further equivalent to $\mathscr{C}$ having enough $\mathbb{E}$-projectives and being hereditary. Assume $\mathrm{Proj}_{\mathbb{E}}(\mathscr{C})^{\wedge}_1 = \mathscr{C}$ holds and let $X \in \mathcal{R}^{\vee}$. Then, there exist an $m \geq 0$ and a sequence of $\mathfrak{s}$-conflations

$$L_0 \to R_0 \to L_1, \quad L_1 \to R_1 \to L_2, \quad \dots, \quad L_{m-1} \to R_{m-1} \to L_m, \quad L_m \to R_m \to 0,$$

where $L_0 = X$ and $R_0, \dots, R_m \in \mathcal{R}$. Since $\mathcal{R}$ is closed under isomorphisms and $R_m \simeq L_m$, it follows that $L_m \in \mathcal{R}$. If $m=0$, $\mathcal{R}$ being closed under direct summands implies $0 \in \mathcal{R}$, and by the $\mathfrak{s}$-conflation $X \to X \to 0$, it follows that $X \in \mathrm{Cocone}(\mathcal{R}, \mathcal{R})$. If $m=1$, then $X \in \mathrm{Cocone}(\mathcal{R}, \mathcal{R})$ by $X \to R_0 \to L_1$. If $m > 1$, since $\mathscr{C}$ is hereditary and $\mathcal{R} \subseteq \mathcal{R}^{\perp_{>0}}$, by iteratively applying general dimension shifting to the $\mathfrak{s}$-conflations we can obtain $L_1 \in \mathcal{R}$ (Exercise!).

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ have $\overbrace{\text{enough } \mathbb{E}\text{-projectives and } \operatorname{pd}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) \leq 1}^{\mathbb{E}^{\geq 2} = 0}$.

Prp. A presilting subcategory $\mathcal{R} \subseteq \mathscr{C}$ is silting if, and only if it is 1-tilting.

Rmk. Under the preceding Setup, we will refer to 1-tilting objects/subcategories simply as *tilting*.
By duality, if $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has enough $\mathbb{E}$-injectives and $\operatorname{id}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) \leq 1$, then silting $\iff$ 1-cotilting; in this latter case, we will refer to 1-cotilting objects/subcategories simply as *cotilting*.

Prp. Tilting subcategories are $\mathbb{E}$-tilting.

Proof.

(c)⇒(d): Suppose $\mathcal{R}$ is tilting, and let $X \in \mathscr{C}$ be such that $\mathbb{E}(X, \mathcal{R}) = 0 = \mathbb{E}(\mathcal{R}, X)$. By our assumptions, there exist $\mathfrak{s}$-conflations $P_1 \to P_0 \to X, P_1 \to R^0 \to R^1$ and $P_0 \to S^0 \to S^1$ with $P_0, P_1 \in \mathrm{Proj}_{\mathbb{E}}(\mathscr{C})$ and $R^0, R^1, S^0, S^1 \in \mathcal{R}$. By [NP19, Proposition 3.15(2)], we have the commutative diagram of $\mathfrak{s}$-conflations in $\mathscr{C}$

$$\begin{tikzcd}
P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & X \arrow[d, equal] \\
R^0 \arrow[r] \arrow[d] & R^0 \oplus X \arrow[r] \arrow[d] & X, \\
R^1 \arrow[r, equal] & R^1 &
\end{tikzcd}$$

where the second row is split since $\mathbb{E}(X, R_0) = 0$. Analogously, we obtain the commutative diagram of $\mathfrak{s}$-conflations in $\mathscr{C}$

Prp. Tilting subcategories are $\mathbb{E}$-tilting.

Proof. (...continued)

$$\begin{tikzcd}
P_0 \arrow[r] \arrow[d] & R^0 \oplus X \arrow[r] \arrow[d] & R^1 \arrow[d, equal] \\
S^0 \arrow[r] \arrow[d] & S^0 \oplus R^1 \arrow[r] \arrow[d] & R^1, \\
S^1 \arrow[r, equal] & S^1 &
\end{tikzcd}$$

where the second row splits because $\mathbb{E}(R^1, S^0) = 0$. Since $\mathcal{R}$ is rigid and $\mathbb{E}(\mathcal{R}, X) = 0$, then $\mathbb{E}(S^1, R^0 \oplus X) \simeq \mathbb{E}(S^1, R^0) \oplus \mathbb{E}(S^1, X) = 0$, by which the $\mathfrak{s}$-conflation $R^0 \oplus X \to S^0 \oplus R^1 \to S^1$ also splits, which implies that $S^0 \oplus R^1 \simeq R^0 \oplus X \oplus S^1$. Since $\mathcal{R}$ is an additive subcategory, then $S^0 \oplus R^1 \in \mathcal{R}$ and, since it is closed under direct summands in $\mathscr{C}$, $X \in \mathcal{R}$.

Rmk. In the dual setup, cotilting $\Rightarrow$ $\mathbb{E}$-tilting.

A rigid subcategory $\mathcal{R} \subseteq \mathscr{C}$ is

silting

enough $\mathbb{E}$-projectives and $\mathbb{E}$-injectives

$+$

$\mathrm{gl.dim}(\mathscr{C},\mathbb{E},\mathfrak{s}) \le 1$

tilting

cotilting

$\mathbb{E}$-tilting

$\Downarrow$

maximal rigid

See [GNP23, Thm. 4.3].

Any $\mathbb{E}$-projective has an $\mathcal{R}$-preenvelope

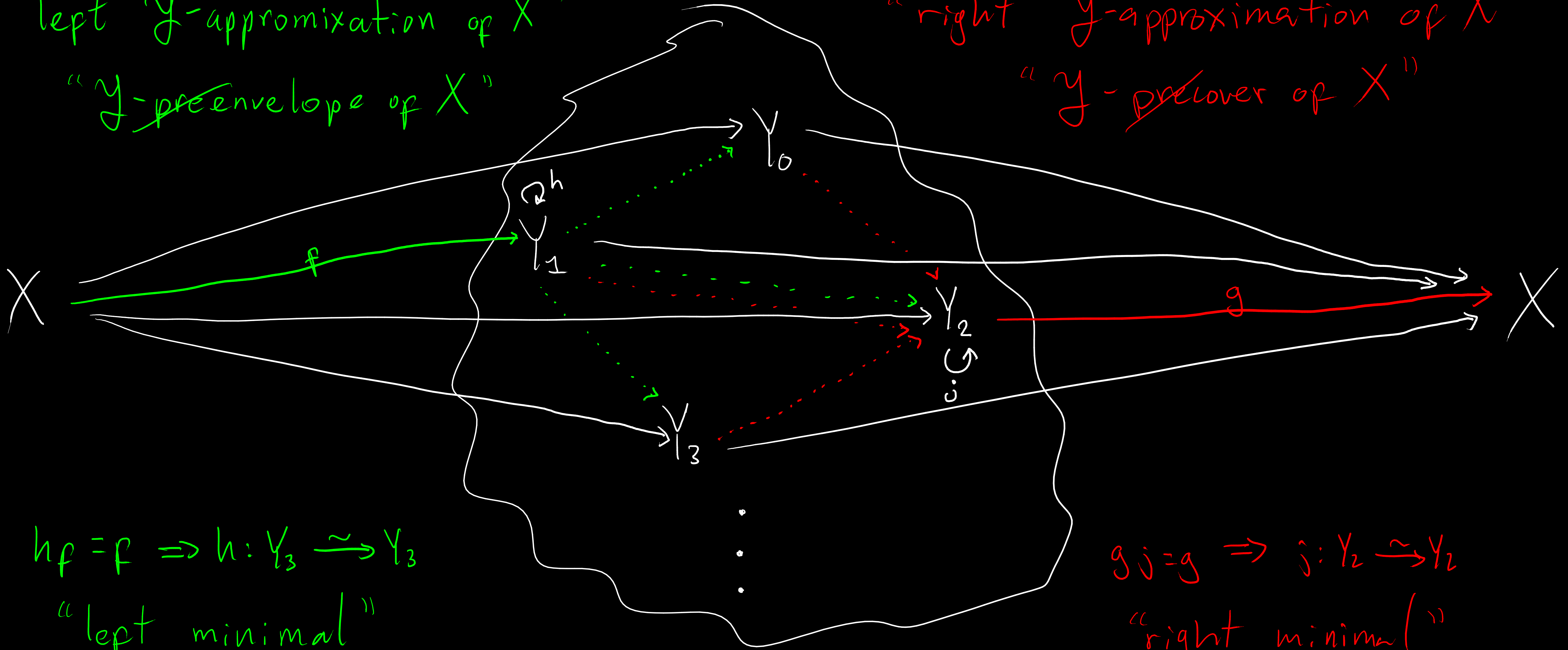
Any $\mathbb{E}$-injective has an $\mathcal{R}$-precover

§2.3 Bongartz (co)completion

$X \xrightarrow{f} Y_3$ $\qquad\qquad$ $\mathcal{Y}$ $\qquad\qquad$ $Y_2 \xrightarrow{g} X$

"left $\mathcal{Y}$-appromixation of X" $\qquad$ "right $\mathcal{Y}$-approximation of X"

"$\mathcal{Y}$-~~pre~~envelope of X" $\qquad$ "$\mathcal{Y}$-~~pre~~cover of X"



$hf=f \Rightarrow h: Y_3 \xrightarrow{\sim} Y_3$ $\qquad$ $gj=g \Rightarrow j: Y_2 \xrightarrow{\sim} Y_2$

"left minimal" $\qquad$ "right minimal"

<u>Dfn</u>. Let $\mathcal{Y} \subseteq \mathcal{C}$ and $C \in \mathcal{C}$.

(1) A $\mathcal{Y}$-*preenvelope* of C is a morphism $C \xrightarrow{f} Y_1$ with $Y_1 \in \mathcal{Y}$ such that, for all $Y' \in \mathcal{Y}$, the induced map $\mathrm{Hom}_{\mathcal{C}}(f, Y') : \mathrm{Hom}_{\mathcal{C}}(Y_1, Y') \to \mathrm{Hom}_{\mathcal{C}}(C, Y')$ is surjective. If, furthermore, for all $h : Y_1 \to Y_1$, $hf = f \Rightarrow h$ is an automorphism, then $C \xrightarrow{f} Y_1$ is a $\mathcal{Y}$-*envelope* of C.

(2) A $\mathcal{Y}$-*precover* of C is a morphism $Y_2 \xrightarrow{g} C$ with $Y_2 \in \mathcal{Y}$ such that, for all $Y' \in \mathcal{Y}$, the induced map $\mathrm{Hom}_{\mathcal{C}}(Y', g) : \mathrm{Hom}_{\mathcal{C}}(Y', Y) \to \mathrm{Hom}_{\mathcal{C}}(Y', C)$ is surjective. If, furthermore, for all $j : Y_2 \to Y_2$, $gj = g \Rightarrow j$ is an automorphism, then $Y_2 \xrightarrow{g} C$ is a $\mathcal{Y}$-*cover* of C.

Stp. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ have "well defined" positive extensions of $\mathbb{E}$.

Dfn. Let R be a presilting object in $\mathcal{C}$. A *Bongartz (co)completion* of R is an object $S \in \mathcal{C}$ such that $\operatorname{add}(R \oplus S) \subseteq \mathcal{C}$ is (co)tilting.

Rmk. When $\mathcal{C}$ is hereditary and there exists $I \in \mathcal{C}$ such that $\operatorname{Inj}_{\mathbb{E}}(\mathcal{C}) = \operatorname{add}(I)$, the Bongartz cocompletion for a rigid (thus, presilting) object $R \in \mathcal{C}$ can be construced as the cocone of an $\operatorname{add}(R)$-precover $R' \xrightarrow{g} I$ of I, for which we need g to be a deflation! Taking inspiration from [PPPP19]...

Setting for Section 4.2. In that section, we let $\mathcal{C}$ be an extriangulated category with a fixed full additive subcategory $\mathcal{T}$, stable under isomorphisms, under taking direct summands, and satisfying the following three properties:

(1) Every $T \in \mathcal{T}$ is projective in $\mathcal{C}$;
(2) For each $T \in \mathcal{T}$, the morphism $T \to 0$ is an inflation for the extriangulated structure of $\mathcal{C}$;
(3) For each $X \in \mathcal{C}$, there is an extriangle $T_1^X \rightarrowtail T_0^X \twoheadrightarrow X \overset{\delta_X}{\dashrightarrow}$ in $\mathcal{C}$ with T_0^X, T_1^X in $\mathcal{T}$.

Lemma 4.33. *Every morphism with domain in $\mathcal{T}$ is an inflation.*

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ have enough $\mathbb{E}$-injectives and $\operatorname{id}(\mathscr{C}, \mathbb{E}, \mathfrak{s}) \leq 1$.

Prp. Suppose that there exists an object $I \in \mathscr{C}$ such that $\operatorname{Inj}_{\mathbb{E}}(\mathscr{C}) = \operatorname{add}(I)$ and $0 \to I$ is a deflation. Let $U \in \mathscr{C}$ be a rigid object. If $U' \to I$ is an $\operatorname{add}(U)$-precover, then it is a deflation and the cocone of the corresponding $\mathfrak{s}$-conflation is the Bongartz cocompletion of U.

Proof. Let $U' \to I$ be an $\operatorname{add}(U)$-precover. By the dual of [PPPP19, Lmm. 4.33], $U' \to I$ is a deflation. Let $P_U \to U' \to I$ be the corresponding conflation, from which we obtain the exact sequences

$$\begin{aligned} \operatorname{Hom}(U, U') \to \operatorname{Hom}(U, I) \to \mathbb{E}(U, P_U) &\to \mathbb{E}(U, U'), \\ \mathbb{E}(U', U) \to \mathbb{E}(P_U, U) &\to \mathbb{E}^2(I, U), \\ \mathbb{E}(U', P_U) \to \mathbb{E}(P_U, P_U) &\to \mathbb{E}^2(I, P_U). \end{aligned}$$

By rigidity of U and biadditivity of $\mathbb{E}$, it follows that $\mathbb{E}(U, U') = 0$ and, since $U' \to I$ being a $\operatorname{add}(U)$-preenvelope implies that $\operatorname{Hom}(U, U') \to \operatorname{Hom}(U, I)$ is surjective, we have that $\mathbb{E}(U, P_U) = 0$. Moreover, by heredity of $\mathscr{C}$ it follows that $\mathbb{E}(P_U, U) = 0 = \mathbb{E}(P_U, P_U)$, which furthermore implies that R_U is rigid. Thus, it follows from the $\mathfrak{s}$-conflation $P_U \to U' \xrightarrow{f} I$ that $\operatorname{add}(R_U)$ is cotilting.

Rmk. Suppose there exists $I \in \mathscr{C}$ such that $\mathrm{Inj}_{\mathbb{E}}(\mathscr{C}) = \mathrm{add}(I)$. Since $\mathscr{C}/\langle \mathrm{Proj}_{\mathbb{E}}(\mathscr{C}) \cap \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})\rangle =: \mathscr{C}'$ is extriangulated, the existence of an $\mathfrak{s}$-conflation $N \to Q \to I$ with $Q \in \mathrm{Proj}_{\mathbb{E}}(\mathscr{C}) \cap \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})$ implies the existence of a deflation $0 \to I$ in $\mathscr{C}'$, and is thus sufficient to apply the previous Proposition to $\mathscr{C}'$. This gives us the following result.

Prp. Suppose that there exists an object $I \in \mathscr{C}$ such that $\mathrm{Inj}_{\mathbb{E}}(\mathscr{C}) = \mathrm{add}(I)$ and there is an $\mathfrak{s}$-conflation $N \to Q \to I$ with $Q \in \mathrm{Proj}_{\mathbb{E}}(\mathscr{C}) \cap \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})$. Let $U \in \mathscr{C}$ be a rigid object. If $U' \to I$ is an $\mathrm{add}(\{U\} \cup (\mathrm{Proj}_{\mathbb{E}}(\mathscr{C}) \cap \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})))$-precover then, up to adding an $\mathbb{E}$-projective-injective summand to U', we may assume it is a deflation. If $P_U \to U' \to I$ is the corresponding $\mathfrak{s}$-conflation, then $\mathrm{add}(\{U \oplus P_U\} \cup (\mathrm{Proj}_{\mathbb{E}}(\mathscr{C}) \cap \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})))$ is cotilting.

Definition 1.2.2.39 [GNP23, Definition 3.5]

(a) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has enough $\mathbb{E}$-projectives, its *dominant dimension*, denoted by $\operatorname{dom.dim}(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, is the largest integer n such that for any $\mathbb{E}$-projective object P there exist n $\mathfrak{s}$-conflations

$$\begin{gathered} P \to Q_0 \to M_1, \\ M_1 \to Q_1 \to M_2, \\ \cdots \\ M_{n-1} \to Q_{n-1} \to M_n, \end{gathered}$$

with Q_k an $\mathbb{E}$-projective-injective for all $0 \leq k \leq n-1$.

(b) If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ has enough $\mathbb{E}$-injectives, its *codominant dimension*, denoted by $\operatorname{codom.dim}(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, is the largest integer n such that for any $\mathbb{E}$-injective object I there exist n $\mathfrak{s}$-conflations

$$\begin{gathered} N_1 \to Q'_0 \to I, \\ N_2 \to Q'_1 \to N_1, \\ \cdots \\ N_n \to Q'_{n-1} \to N_{n-1}, \end{gathered}$$

with Q'_k an $\mathbb{E}$-projective-injective for all $0 \leq k \leq n-1$.

§2.4 0-Auslander extriangulated categories

Rmk. Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ have enough $\mathbb{E}$-projectives, $\operatorname{pd}(\mathscr{C},\mathbb{E},\mathfrak{s})\leq 1\leq \operatorname{dom.dim}(\mathscr{C},\mathbb{E},\mathfrak{s})$, and an object $P\in\mathscr{C}$ such that $\operatorname{Proj}_{\mathbb{E}}(\mathscr{C})=\operatorname{add}(P)$. Then, by duality, any rigid object $R\in\mathscr{C}$ for which there exists an $\operatorname{add}(\{R\}\cup(\operatorname{Proj}_{\mathbb{E}}(\mathscr{C})\cap\operatorname{Inj}_{\mathbb{E}}(\mathscr{C})))$-preenvelope of P has a Bongartz completion.

By [GNP23], we have the following equivalence.

Proposition 3.6. *Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category. The following conditions are equivalent.*

(i) $(\mathscr{C},\mathbb{E},\mathfrak{s})$ *has enough projectives and* $\operatorname{pd}(\mathscr{C},\mathbb{E})\leq 1\leq \operatorname{dom.dim}(\mathscr{C},\mathbb{E})$.
(ii) $(\mathscr{C},\mathbb{E},\mathfrak{s})$ *has enough injectives and* $\operatorname{id}(\mathscr{C},\mathbb{E})\leq 1\leq \operatorname{codom.dim}(\mathscr{C},\mathbb{E})$.

This motivates our main definition.

Dfn. An extriangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$ is *0-Auslander* if it has enough $\mathbb{E}$-projectives and $\mathrm{pd}(\mathscr{C},\mathbb{E},\mathfrak{s})\leq 1\leq \mathrm{dom.dim}(\mathscr{C},\mathbb{E},\mathfrak{s})$ (equiv., if it has enough $\mathbb{E}$-injectives and $\mathrm{pd}(\mathscr{C},\mathbb{E},\mathfrak{s})\leq 1\leq \mathrm{codom.dim}(\mathscr{C},\mathbb{E},\mathfrak{s})$).

Rmk. Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be 0-Auslander. By the preceding discussion, we have the following.

(1) tilting $\Leftrightarrow$ silting $\Leftrightarrow$ cotilting $\Rightarrow$ $\mathbb{E}$-tilting $\Rightarrow$ maximal rigid.

(2) If $\mathrm{Proj}_{\mathbb{E}}(\mathscr{C})=\mathrm{add}(P)$ for some $P\in\mathscr{C}$, any rigid object $R\in\mathscr{C}$ for which there exists an $\mathrm{add}(\{R\}\cup(\mathrm{Proj}_{\mathbb{E}}(\mathscr{C})\cap\mathrm{Inj}_{\mathbb{E}}(\mathscr{C})))$-preenvelope of P has a Bongartz completion. Moreover, the dual result also holds.

Exp.

(1) For a basic connected f.d. $\mathbb{K}$-alg. with $\mathbb{K}=\overline{\mathbb{K}}$, $\mathrm{mod}(\Lambda)$ is *0-Auslander* iff Λ is semisimple or (Morita equivalent to) the path algebra of $\mathbb{K}\vec{A}_n$ for some $n\geq 1$.

(2) However, for an Artin algebra Λ, $K_\Lambda := K^{[-1,0]}(\mathrm{proj}(\Lambda))$ is *reduced 0-Auslander*! It is also KS/HF.

- projectives: $\mathrm{Hom}_{D(\Lambda)}(P, X[1]) = \mathbb{E}(P,X) = 0 \ \forall X \in K_\Lambda \iff P = \begin{smallmatrix} 0 \\ \downarrow \\ P \end{smallmatrix}$ (dually, $I \in \mathrm{Inj}(K_\Lambda) \iff I = \begin{smallmatrix} P \\ \downarrow \\ 0 \end{smallmatrix} = P[1]$, $P \in \mathrm{Proj}(K_\Lambda)$)

- enough projectives & $\mathrm{pd}(K_\Lambda) \leq 1$: $\forall$ $X = \begin{smallmatrix} P^{-1} \\ p\downarrow \\ P^0 \end{smallmatrix} \in K_\Lambda$, we have the distinguished triangle $\begin{smallmatrix} 0 \\ \downarrow \\ P^0 \end{smallmatrix} \to \begin{smallmatrix} P^{-1} \\ p\downarrow \\ P^0 \end{smallmatrix} \to \begin{smallmatrix} P^{-1} \\ \downarrow \\ 0 \end{smallmatrix} \xrightarrow{p} \begin{smallmatrix} P^0 \\ \downarrow \\ 0 \end{smallmatrix}$ in $D(\Lambda)$, which we can rotate to $\begin{smallmatrix} 0 \\ \downarrow \\ P^{-1} \end{smallmatrix} \xrightarrow{p} \begin{smallmatrix} 0 \\ \downarrow \\ P^0 \end{smallmatrix} \to \begin{smallmatrix} P^{-1} \\ p\downarrow \\ P^0 \end{smallmatrix} \to \begin{smallmatrix} P^{-1} \\ \downarrow \\ 0 \end{smallmatrix}$, thus obtaining the conflation $P^{-1} \to P^0 \to X$ in K_Λ.

- $\mathrm{dom.dim}(K_\Lambda) \geq 1$: $\forall P = \begin{smallmatrix} 0 \\ \downarrow \\ P \end{smallmatrix} \in \mathrm{Proj}(K_\Lambda)$, we have the distinguished triangle $P \xrightarrow{1_P} P \to 0 \to P[1]$ in $D(\Lambda)$, which we can rotate to $P \to 0 \to P[1] \xrightarrow{1_{P[1]}} P[1]$, thus obtaining the conflation $P \to 0 \to P[1]$ in K_Λ.

- reduced: $\mathrm{Proj}(K_\Lambda) \cap \mathrm{Inj}(K_\Lambda) = \left\{ \begin{smallmatrix} 0 \\ \downarrow \\ P \end{smallmatrix} \,\middle|\, P \in \mathrm{proj}(\Lambda) \right\} \cap \left\{ \begin{smallmatrix} P \\ \downarrow \\ 0 \end{smallmatrix} \,\middle|\, P \in \mathrm{proj}(\Lambda) \right\} = \left\{ \begin{smallmatrix} 0 \\ \downarrow \\ 0 \end{smallmatrix} \right\}$.

Moreover, $K^{[-1,0]}(\mathrm{Proj}(A))$ and its dual are *r0-A* for any ring A, though they may not be KS/ES/HF.

(3) Let $(\mathcal{T},[1],\Delta)$ be a HF R-linear triangulated category for some ring R. A pair $(\mathcal{X},\mathcal{Y})$ of full subcategories of $\mathcal{T}$ such that

(i) $\mathcal{X} = {}^{\perp_0}\mathcal{Y} := \{X\in\mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X,Y)=0\ \forall Y\in\mathcal{Y}\}$ and, dually, $\mathcal{X}^{\perp_0}=\mathcal{Y}$;

(ii) $\mathcal{T}=\mathcal{X}*\mathcal{Y}$: $\forall C\in\mathcal{T}$ there exists $X\to C\to Y\to X[1]$ in Δ with $X\in\mathcal{X}, Y\in\mathcal{Y}$;

$\Big\}$ (complete torsion pair)

(iii) $\mathcal{X}\subseteq\mathcal{X}[1]$ (equiv. $\mathcal{Y}[1]\subseteq\mathcal{Y}$);

is a *co-t-structure*, its *coheart* is $\mathcal{S}:=\mathcal{X}[1]\cap\mathcal{Y}$ and its *extended coheart* is $\mathcal{C}:=\mathcal{S}*\mathcal{S}[1]=\mathcal{X}[2]\cap\mathcal{Y}$. Since $\mathcal{C}$ is closed under extensions in $\mathcal{T}$, it has an extriangulated structure induced by $\mathrm{Ext}^1_{\mathcal{T}}(-,?)=\mathrm{Hom}_{\mathcal{T}}(-,?[1])$ and Δ (similarly to $K^{[-1,0]}(\mathrm{proj}(\Lambda))\underset{\text{ext.}}{\subseteq}K^b(\mathrm{proj}(\Lambda))$). If $S\in\mathcal{S}$ then, $\forall C\in\mathcal{C}$, $\mathbb{E}(S,C)=\mathrm{Hom}_{\mathcal{T}}(S,C[1])=0$ since $S\in\mathcal{X}[1]$, $C\in\mathcal{Y}[1]$ and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}[1],\mathcal{Y}[1])\simeq\mathrm{Hom}_{\mathcal{T}}(\mathcal{X},\mathcal{Y})=0$ by which $\mathcal{S}\subseteq\mathrm{Proj}_{\mathbb{E}}(\mathcal{C})$. Thus, $\forall C\in\mathcal{C}=\mathcal{S}*\mathcal{S}[1]$, we can rotate the corresponding triangle $S\to C\to S'[1]\to S[1]$ in Δ to obtain a projective resolution of C in $\mathcal{C}$, proving $\mathcal{C}$ has enough projectives. Since $\mathbb{E}^2(\mathcal{C},\mathcal{C})=\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}[2]\cap\mathcal{Y},\mathcal{X}[4]\cap\mathcal{Y}[2])\subseteq\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}[2],\mathcal{Y}[2])\simeq\mathrm{Hom}_{\mathcal{T}}(\mathcal{X},\mathcal{Y})$, $\mathcal{C}$ is hereditary, which is equivalent to $\mathrm{pd}(\mathcal{C})\leq 1$. Moreover, $\mathrm{dom.dim}(\mathcal{C})\geq 1$ is proved as in (2). Thus, extended cohearts of the co-t-structures are 0-Auslander.

(4) HF, 2CY $\mathbb{K}$-linear triangulated categories with a cluster tilting object (e.g. cluster categories of acyclic quivers or of Jacobi-finite quivers with potentials) are 0-Auslander.

(5) The "subcategory of walks" of $\mathrm{mod}(A^{✿})$, the category of finitely generated modules over the path algebra of the blossoming of a gentle bound quiver, is 0-Auslander.

(6) Rigid subcategories of extriangulated categories $\longrightarrow$ relative tilting theory

(via the maximal relative extriangulated structure making all objects in $\mathcal{R}$ projective)

triangulated

exact

- Triangulated categories with a cluster-tilting object
- (3)

- $\mathcal{C}^{[-1,0]}(\mathcal{A})$
- Frobenius exact cluster categories

and more!

DISCLAIMER: I do not actually understand examples (4)-(6)!

§3. Irreducible mutation in 0-Auslander extriangulated categories

§3.1 Rigid reduction

Rmk. If $\mathscr{D} \subseteq \mathscr{C}$ is a full additive subcategory closed under isomorphisms and extensions in $\mathscr{C}$, it follows that $(\mathscr{D}, \mathbb{E}_{\mathscr{D}}, \mathfrak{s}_{\mathscr{D}})$ is an extriangulated category, where $\mathbb{E}_{\mathscr{D}}$ is the restriction of $\mathbb{E}$ to $\mathscr{D}^{\mathrm{op}} \times \mathscr{D}^{\mathrm{op}}$ and $\mathfrak{s}_{\mathscr{D}}$ is the restriction of $\mathfrak{s}$ to $\mathbb{E}_{\mathscr{D}}$, which we call the *restriction* of $\mathscr{C}$ to $\mathscr{D}$.

Dfn. For $\mathcal{R} \subseteq \mathscr{C}$, we define ${}^{\perp_1}\mathcal{R} := \{X \in \mathscr{C} \mid \mathbb{E}(X,R)=0 \ \forall R \in \mathcal{R}\}$ and $\mathcal{R}^{\perp_1} := \{Y \in \mathscr{C} \mid \mathbb{E}(R,Y)=0 \ \forall R \in \mathcal{R}\}$.

Remark 1.2.2.21 Let $\mathcal{R} \subseteq \mathscr{C}$ be rigid and closed under direct summands in $\mathscr{C}$. Then, the subcategory ${}^{\perp_1}\mathcal{R} \cap \mathcal{R}^{\perp_1} \subseteq \mathscr{C}$ is additive and closed under direct summands and extensions in $\mathscr{C}$. Then, it follows that the restriction $({}^{\perp_1}\mathcal{R} \cap \mathcal{R}^{\perp_1}, \mathbb{E}_{{}^{\perp_1}\mathcal{R} \cap \mathcal{R}^{\perp_1}}, \mathfrak{s}_{{}^{\perp_1}\mathcal{R} \cap \mathcal{R}^{\perp_1}})$ is an extriangulated category. Note that $\mathcal{R} \subseteq {}^{\perp_1}\mathcal{R} \cap \mathcal{R}^{\perp_1}$ and, moreover, all of the elements of $\mathcal{R}$ are $\mathbb{E}_{{}^{\perp_1}\mathcal{R} \cap \mathcal{R}^{\perp_1}}$-projective-injective in ${}^{\perp_1}\mathcal{R} \cap \mathcal{R}^{\perp_1}$. Thus, we obtain the reduced extriangulated category

$$ {}^{\perp_1}\mathcal{R} \cap \mathcal{R}^{\perp_1} / \langle \mathcal{R} \rangle =: \overline{\mathscr{C}}_{\mathcal{R}}, $$

which we denote explicitely by $(\overline{\mathscr{C}}_{\mathcal{R}}, \overline{\mathbb{E}}, \overline{\mathfrak{s}})$ and call the *reduction* of $\mathscr{C}$ by $\mathcal{R}$. We note that, if $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is hereditary, then $(\overline{\mathscr{C}}_{\mathcal{R}}, \overline{\mathbb{E}}, \overline{\mathfrak{s}})$ is also hereditary.

§3.2 Almost complete rigidity

Definition 4.1.2.6 A rigid subcategory $\mathcal{R}' \subseteq \mathscr{C}$ is *almost complete* if

- it is not (co)tilting;
- it contains all $\mathbb{E}$-projective-injective objects;
- there exists an indecomposable object $R \in \mathscr{C}$ such that $\operatorname{add}(\mathcal{R}' \cup \{R\})$ is (co)tilting;

in which case R is called a *complement* for $\mathcal{R}'$.

Thm. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be reduced 0-Auslander and $\mathcal{R}' \subseteq \mathscr{C}$ be an almost complete rigid subcategory such that
(i) all $\mathbb{E}$-projectives have an $\mathcal{R}'$-preenvelope and, dually, all $\mathbb{E}$-injectives have an $\mathcal{R}'$-precover;
(ii) the reduction $(\overline{\mathscr{C}}_{\mathcal{R}'}, \overline{\mathbb{E}}, \overline{\mathfrak{s}})$ of $\mathscr{C}$ by $\mathcal{R}'$ is a Krull-Schmidt category;
(iii) if X is a complement for $\mathcal{R}'$, then any $\overline{\mathbb{E}}$-projective has an $\operatorname{add}(X)$-preenvelope in $\overline{\mathscr{C}}_{\mathcal{R}'}$ (equiv., dual).

Then there exist precisely two complements for $\mathcal{R}'$ in $\mathscr{C}$ up to isomorphism.

Thm. Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be reduced 0-Auslander and $\mathcal{R}'\subseteq\mathscr{C}$ be an almost complete rigid subcategory such that
(i) all $\mathbb{E}$-projectives have an $\mathcal{R}'$-preenvelope and, dually, all $\mathbb{E}$-injectives have an $\mathcal{R}'$-precover;
(ii) the reduction $(\overline{\mathscr{C}}_{\mathcal{R}'},\overline{\mathbb{E}},\overline{\mathfrak{s}})$ of $\mathscr{C}$ by $\mathcal{R}'$ is a Krull-Schmidt category;
(iii) if X is a complement for $\mathcal{R}'$, then any $\overline{\mathbb{E}}$-projective has an $\operatorname{add}(X)$-preenvelope in $\overline{\mathscr{C}}_{\mathcal{R}'}$ (equiv., dual).

Then there exist precisely two complements for $\mathcal{R}'$ in $\mathscr{C}$ up to isomorphism.

Rmk. Condition (i) is satisfied if $\mathcal{R}'$ has finitely many indecomposables up to isomorphism, whereas condition (iii) is satisfied if $\mathscr{C}$ is Hom-finite.

We will divide the proof into three steps:

1. Establish an "exchange conflation".
2. Show that $\mathcal{R}'$ has at most two complements.
3. Show that $\mathcal{R}'$ has at least two complements.

§3.3 Exchange conplations

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be reduced 0-Auslander.

Prp. Let $\mathcal{U} \subseteq \mathscr{C}$ be a rigid subcategory closed under direct summands and containing all projective-injective objects. Suppose any $I \in \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})$ has a $\mathcal{U}$-precover, denoted by f^I, and denote its corresponding cocone by $P^I_{\mathcal{U}}$. Then $\mathrm{add}(\mathcal{U} \oplus (\bigoplus_{I \in \mathrm{Inj}_{\mathbb{E}}(\mathscr{C})} P^I_{\mathcal{U}}))$ is a cosilting subcategory of $\mathscr{C}$.

Proof. Analogous to the previous proof of Bongartz cocompletion.

Lmm. Let $X \to 0 \to Y$ be a conflation. Then X is projective, Y is injective, and both are rigid.

Proof. Exercise! (Hint: Use extended long exact sequences.)

Rmk. By [NP19, 3.15], as a direct consequence of (ET4), pulling back along a pair of deflations with the same codomain we can obtain two conflations with the same extension:

$$\begin{tikzcd}
 & A' \arrow[r, equal] \arrow[d, dotted] & A' \arrow[d, "x'"] \\
A \arrow[r, dotted] \arrow[d, equal] & E \arrow[r] \arrow[d] & B' \arrow[d, "g'"] \\
A \arrow[r, "x"'] & B \arrow[r, "y"'] & C.
\end{tikzcd}$$

Dually, by (ET4)*, pushing out along a pair of inflations with the same domain produces two conflations with the same extension:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f'"'] & B \arrow[r, "g"] \arrow[d] & C \arrow[d, equal] \\
B' \arrow[r] \arrow[d, "g'"'] & F \arrow[r, dotted] \arrow[d, dotted] & C. \\
C' \arrow[r, equal] & C' &
\end{tikzcd}$$

Due to their importance, these diagrams are known as "shifted octahedra".

Prp. Let $\mathcal{R}' \subseteq \mathcal{C}$ be an almost complete rigid subcategory satisfying (i). Then, for any indecomposable $I \in \mathrm{Inj}_{\mathbb{E}}(\mathcal{C})$, there exist $\mathfrak{s}$-conflations $P \to 0 \to I$, $Q \to R \to I$ and $P \to R' \to J$ such that $P \in \mathrm{Proj}_{\mathbb{E}}(\mathcal{C})$, $R, R' \in \mathcal{R}'$, $\mathrm{add}(\mathcal{R}' \cup \{Q\})$ and $\mathrm{add}(\mathcal{R}' \cup \{J\})$ are rigid. Moreover, we obtain an $\bar{\mathfrak{s}}$-conflation $Q \longrightarrow 0 \longrightarrow J$ with $Q, J \neq 0$ in the reduction $(\overline{\mathcal{C}}_{\mathcal{R}'}, \overline{\mathbb{E}}, \bar{\mathfrak{s}})$ of $\mathcal{C}$ by $\mathcal{R}'$.

Proof. Let I be an indecomposable $\mathbb{E}$-injective which does not belong to $\mathcal{R}'$, which must exist because otherwise $\mathcal{R}'$ would be a cotilting subcategory. Since $\mathrm{codom.dim}(\mathcal{C}, \mathbb{E}, \mathfrak{s}) \geq 1$, there exists a conflation $P \to G \to I$ with $G \in \mathrm{Proj}_{\mathbb{E}}(\mathcal{C}) \cap \mathrm{Inj}_{\mathbb{E}}(\mathcal{C}) = \{0\}$. Thus, $P \in \mathrm{Proj}_{\mathbb{E}}(\mathcal{C})$ by the previous Lemma. By (i), I has an $\mathcal{R}'$-precover $R \to I$ and P has an $\mathcal{R}'$-preenvelope $P \to R'$. Proceding as in the proof of Bongartz cocompletion (and its dual) we see that $R \to I$ is a deflation, $P \to R'$ is an inflation and, if the corresponding conflations are $Q \to R \to I$ and $P \to R' \to J$, then $\mathrm{add}(\mathcal{R}' \cup \{Q\})$ and $\mathrm{add}(\mathcal{R}' \cup \{J\})$ are (co)tilting. Now, by the shifted octahedra, we obtain the following diagrams of $\mathfrak{s}$-conflations:

Prp. Let $\mathcal{R}' \subseteq \mathcal{C}$ be an almost complete rigid subcategory satisfying (i). Then, for any indecomposable $I \in \operatorname{Inj}_{\mathbb{E}}(\mathcal{C})$, there exist $\mathfrak{s}$-conflations $P \to O \to I$, $Q \to R \to I$ and $P \to R' \to J$ such that $P \in \operatorname{Proj}_{\mathbb{E}}(\mathcal{C})$, $R, R' \in \mathcal{R}$, $\operatorname{add}(\mathcal{R}' \cup \{Q\})$ and $\operatorname{add}(\mathcal{R}' \cup \{J\})$ are rigid. Moreover, we obtain an $\bar{\mathfrak{s}}$-conflation $Q \longrightarrow O \longrightarrow J$ with $Q, J \neq O$ in the reduction $(\overline{\mathcal{C}}_{\mathcal{R}'}, \overline{\mathbb{E}}, \bar{\mathfrak{s}})$ of $\mathcal{C}$ by $\mathcal{R}'$.

Proof. (...continued)

$$\begin{tikzcd} & P \arrow[r, equal] \arrow[d] & P \arrow[d] \\ Q \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & O \arrow[d] \\ Q \arrow[r] & R \arrow[r] & I \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d] & J \arrow[d, equal] \\ Q \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & J \\ R \arrow[r, equal] & R & \end{tikzcd},$$

where $E \cong R \oplus R'$, since $R, R' \in \mathcal{R}$, which is rigid. Thus, $Q \to R \oplus R' \to J$ descends to $Q \to O \to J$ in $\overline{\mathcal{C}}_{\mathcal{R}'}$. Now, since $I \notin \mathcal{R}'$, then $Q \neq O$ by the $\mathfrak{s}$-conflation $Q \to R \to I$. Moreover, $Q \notin \mathcal{R}'$ since otherwise $\mathcal{R}'$ would be cotilting. Thus, $Q \neq O$ in $\overline{\mathcal{C}}_{\mathcal{R}'}$, which implies $J \neq O$ in $\overline{\mathcal{C}}_{\mathcal{R}'}$ by the $\bar{\mathfrak{s}}$-conflation $Q \to O \to J$. ∎

§3.4 At most two complements

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be reduced 0-Auslander and $\mathcal{R}' \subseteq \mathscr{C}$ be an almost complete rigid subcategory such that
(i) all $\mathbb{E}$-projectives have an $\mathcal{R}'$-preenvelope and, dually, all $\mathbb{E}$-injectives have an $\mathcal{R}'$-precover;
(ii) the reduction $(\overline{\mathscr{C}}_{\mathcal{R}'}, \overline{\mathbb{E}}, \bar{\mathfrak{s}})$ of $\mathscr{C}$ by $\mathcal{R}'$ is a Krull-Schmidt category;
(iii) if X is a complement for $\mathcal{R}'$, then any $\overline{\mathbb{E}}$-projective has an $\mathrm{add}(X)$-preenvelope in $\overline{\mathscr{C}}_{\mathcal{R}'}$ (equiv., dual).

Lemma 4.23. *Let $\mathscr{C}$ be an extriangulated category, let $\mathcal{R}$ be a Krull–Schmidt, rigid subcategory, let $P \in \mathscr{C}$ be projective, and let $P \rightarrowtail R^0 \twoheadrightarrow R^1 \dashrightarrow$ be an $\mathfrak{s}$-triangle with $R^0, R^1 \in \mathcal{R}$ and where the inflation is left minimal.* Then $\mathrm{add}\, R^0 \cap \mathrm{add}\, R^1 = 0$.

Proof. See [GNP23].

§3.4 At most two complements

Prp. $\mathcal{R}'$ has at most two complements.

Proof. Let X be a complement for $\mathcal{R}'$. Then, in particular, X is rigid in $\mathcal{C}$ and, further, in $\overline{\mathcal{C}}_{\mathcal{R}'}$. By Section 3.3 we have an inflation $Q \to 0$ in $\overline{\mathcal{C}}_{\mathcal{R}'}$, with $Q \in \mathrm{Proj}_{\overline{\mathbb{E}}}(\overline{\mathcal{C}}_{\mathcal{R}'})$. Thus by [PPPP19, Lmm. 4.33], all morphisms in $\overline{\mathcal{C}}_{\mathcal{R}'}$ with domain Q are inflations. By condition (iii) in our Setup, Q has an $\mathrm{add}(X)$-preenvelope in $\overline{\mathcal{C}}_{\mathcal{R}'}$ $Q \xrightarrow{i} X_0$. Moreover, since $\overline{\mathcal{C}}_{\mathcal{R}'}$ is Krull-Schmidt, we can restrict i along a direct summand of X_0 such that it becomes left minimal; thus, without loss of generality, we can assume that $Q \xrightarrow{i} X_0$ is an $\mathrm{add}(X)$-envelope of Q, which is also an inflation since it has domain Q. Let $Q \xrightarrow{i} X_0 \to Y$ be the corresponding $\overline{\mathfrak{s}}$-conflation, which induces the exact sequences

$$\overbrace{\cancel{\overline{\mathbb{E}}(X,X_0)}}^{\text{rigidity} \to 0} \to \overline{\mathbb{E}}(X,Y) \to \overbrace{\cancel{\overline{\mathbb{E}}^2(X,Q)}}^{\text{heredity} \to 0},$$

$$\mathrm{Hom}_{\overline{\mathcal{C}}_{\mathcal{R}'}}(X_0,X) \xrightarrow{\mathrm{Hom}_{\overline{\mathcal{C}}_{\mathcal{R}'}}(i,X)} \mathrm{Hom}_{\overline{\mathcal{C}}_{\mathcal{R}'}}(Q,X) \to \overbrace{\cancel{\overline{\mathbb{E}}(Y,X)}}^{\text{surjectivity} \to 0} \to \overbrace{\cancel{\overline{\mathbb{E}}(X_0,X)}}^{\text{rigidity} \to 0}.$$

§3.4 At most two complements

Prp. $\mathcal{R}'$ has at most two complements.

Proof. (... continued) Thus, $\overline{\mathbb{E}}(X,Y)=0=\overline{\mathbb{E}}(Y,X)$. Now, by hypothesis, $\operatorname{add}(\mathcal{R}'\cup\{X\})$ is (co)tilting, by which it is $\mathbb{E}$-tilting and, moreover, X is $\overline{\mathbb{E}}$-tilting. Thus, $Y\in\operatorname{add}(X)$. By the $\overline{\mathfrak{s}}$-conflation $Q\to X_0\to Y$ with $Q\in\operatorname{Proj}_{\overline{\mathbb{E}}}(\overline{\mathcal{C}}_{\mathcal{R}'})$ and the previous Lemma, we have that $X_0=0$ or $Y=0$ but not both, since $Q\neq 0$ by Section 3.3.

If $Y=0$, then $Q\to X_0\to 0$ realizes $\delta\in\overline{\mathbb{E}}(0,Q)=\{0\}$, by which $X_0\simeq Q\oplus 0=Q$ and, in particular, $X_0\in\operatorname{Proj}_{\overline{\mathbb{E}}}(\overline{\mathcal{C}}_{\mathcal{R}'})$. Since $X_0\in\operatorname{add}(X)$ and X is indecomposable, we also have $X\in\operatorname{add}(X_0)$, by which $X\in\operatorname{Proj}_{\overline{\mathbb{E}}}(\overline{\mathcal{C}}_{\mathcal{R}'})$. Thus, for all $P\in\operatorname{Proj}_{\overline{\mathbb{E}}}(\overline{\mathcal{C}}_{\mathcal{R}'})$, we have that $\overline{\mathbb{E}}(X,P)=0=\overline{\mathbb{E}}(P,X)$, which implies $P\in\operatorname{add}(X)\subseteq\overline{\mathcal{C}}_{\mathcal{R}'}$ due to X being an $\overline{\mathbb{E}}$-tilting object. This means that X is an additive generator of $\operatorname{Proj}_{\overline{\mathbb{E}}}(\overline{\mathcal{C}}_{\mathcal{R}'})$. Moreover, since X is indecomposable, it is the unique indecomposable $\overline{\mathbb{E}}$-projective object in $\overline{\mathcal{C}}_{\mathcal{R}'}$.

§3.4 At most two complements

Prp. $\mathcal{R}'$ has at most two complements.

Proof. (...continued) On the other hand, if $X_0 = 0$, then by the $\bar{\mathfrak{s}}$-conflation $Q \to X_0 \to Y$ and the Lemma in Section 3.3, Y is $\bar{\mathbb{E}}$-injective. Since $Y \in \operatorname{add}(X)$ and $\bar{\mathbb{E}}$-injectives are closed under direct summands, $X \in \operatorname{Inj}_{\bar{\mathbb{E}}}(\bar{\mathcal{C}}_{\mathcal{R}'})$. Moreover, due to X being indecomposable then, by dual arguments to the preceding case, X is the unique indecomposable $\bar{\mathbb{E}}$-injective object in $\bar{\mathcal{C}}_{\mathcal{R}'}$.

In conclusion, there are at most two scenarios under which an indecomposable object X can be a complement for $\mathcal{R}'$ and, in each one of them, X turns out to be an object which is unique to that scenario. Thus, $\mathcal{R}'$ has at most two complements. ∎

§3.5 At least two complements

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be reduced 0-Auslander and $\mathscr{R}' \subseteq \mathscr{C}$ be an almost complete rigid subcategory such that

(i) all $\mathbb{E}$-projectives have an $\mathscr{R}'$-preenvelope and, dually, all $\mathbb{E}$-injectives have an $\mathscr{R}'$-precover;

(ii) the reduction $(\overline{\mathscr{C}}_{\mathscr{R}'}, \overline{\mathbb{E}}, \bar{\mathfrak{s}})$ of $\mathscr{C}$ by $\mathscr{R}'$ is a Krull-Schmidt category;

(iii) if X is a complement por $\mathscr{R}'$, then any $\overline{\mathbb{E}}$-projective has an $\mathrm{add}(X)$-preenvelope in $\overline{\mathscr{C}}_{\mathscr{R}'}$ (equiv., dual).

Prp. $\mathscr{R}'$ has at least two complements.

Proof. Pending review!

§ 3.6 Irreducible mutations

Thm. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be reduced 0-Auslander and $\mathcal{R}' \subseteq \mathcal{C}$ be an almost complete rigid subcategory such that

(i) all $\mathbb{E}$-projectives have an $\mathcal{R}'$-preenvelope and, dually, all $\mathbb{E}$-injectives have an $\mathcal{R}'$-precover;

(ii) the reduction $(\overline{\mathcal{C}}_{\mathcal{R}'}, \overline{\mathbb{E}}, \bar{\mathfrak{s}})$ of $\mathcal{C}$ by $\mathcal{R}'$ is a Krull-Schmidt category;

(iii) if X is a complement for $\mathcal{R}'$, then any $\overline{\mathbb{E}}$-projective has an $\operatorname{add}(X)$-preenvelope in $\overline{\mathcal{C}}_{\mathcal{R}'}$ (equiv., dual).

Then there exist precisely two complements for $\mathcal{R}'$ in $\mathcal{C}$ up to isomorphism.

Proof. It follows from the main Propositions in Sections 3.3, 3.4 & 3.5. ∎

Cor. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be reduced 0-Auslander, $\mathcal{R} \subseteq \mathscr{C}$ be silting, $R \in \mathcal{R}$ be indecomposable and non-projective-injective, and $\mathcal{R}' = \mathrm{add}(\{S \in \mathcal{R},\ S \text{ is indecomposable and } S \not\cong R\})$. Then, there exist

- an indecomposable $R^* \not\cong R$ unique up to isomorphism such that $\mathrm{add}(\mathcal{R}' \cup \{R^*\})$ is silting;
- an exchange conflation $R \to R' \to R^*$ with $R' \in \mathcal{R}$ or $R^* \to R'' \to R$ with $R'' \in \mathcal{R}$, but not both.

Rmk. Applying this result to the previous examples allows us to obtain or recovery theories of mutation for

(1) maximal almost rigid modules in type A;

(2) 2-term silting complexes (small);

(3) intermediate co-t-structures;

(4) cluster-tilting objects;

(5) non-kissing walks;

(6) relative tilting objects;

(7) functorially finite torsion classes and support τ-tilting subcategories.

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